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ΒΙΒΛΙΟΘΗΚΗ
Μ. ΘΑΛΑΤΤΗΣ

ΒΙΒΛΙΟΘΗΚΗ
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BY

GEORGE A. WENTWORTH

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PLANE AND SOLID

GEOMETRY

BY

G. A. WENTWORTH
AUTHOR OF A SERIES OF TEXT-BOOKS IN MATHEMATICS

REVISED EDITION

ΒΙΒΛΙΟΘΗΚΗ
Μ. ΘΑΛΑΤΤΗΣ

BOSTON, U.S.A.
GINN & COMPANY, PUBLISHERS
The Athenæum Press
1902

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ΒΙΒΛΙΟΘΗΚΗ
Μ. ΘΑΛΑΤΤΗΣ

PREFACE.

Most persons do not possess, and do not easily acquire, the power of abstraction requisite for apprehending geometrical conceptions, and for keeping in mind the successive steps of a continuous argument. Hence, with a very large proportion of beginners in Geometry, it depends mainly upon the *form* in which the subject is presented whether they pursue the study with indifference, not to say aversion, or with increasing interest and pleasure.

Great care, therefore, has been taken to make the pages attractive. The figures have been carefully drawn and placed in the middle of the page, so that they fall directly under the eye in immediate connection with the text; and in no case is it necessary to turn the page in reading a demonstration. Full, long-dashed, and short-dashed lines of the figures indicate given, resulting, and auxiliary lines, respectively. Bold-faced, italic, and roman type has been skilfully used to distinguish the hypothesis, the conclusion to be proved, and the proof.

As a further concession to the beginner, the reason for each statement in the early proofs is printed in small italics, immediately following the statement. This prevents the necessity of interrupting the logical train of thought by turning to a previous section, and compels the learner to become familiar with a large number of geometrical truths by constantly seeing and repeating them. This help is gradually discarded, and the pupil is left to depend upon the knowledge already acquired, or to find the reason for a step by turning to the given reference.

It must not be inferred, because this is not a geometry of interrogation points, that the author has lost sight of the real object of the study. The training to be obtained from carefully following the logical steps of a complete proof has been provided for by the Propositions of the

Geometry, and the development of the power to grasp and prove new truths has been provided for by original exercises. The chief value of any Geometry consists in the happy combination of these two kinds of training. The exercises have been arranged according to the test of experience, and are so abundant that it is not expected that any one class will work them all out. The methods of attacking and proving original theorems are fully explained in the first Book, and illustrated by sufficient examples; and the methods of attacking and solving original problems are explained in the second Book, and illustrated by examples worked out in full. None but the very simplest exercises are inserted until the student has become familiar with geometrical methods, and is furnished with elementary but much needed instruction in the art of handling original propositions; and he is assisted by diagrams and hints as long as these helps are necessary to develop his mental powers sufficiently to enable him to carry on the work by himself.

The law of converse theorems, the distinction between positive and negative quantities, and the principles of reciprocity and continuity have been briefly explained; but the application of these principles is left mainly to the discretion of teachers.

The author desires to express his appreciation of the valuable suggestions and assistance which he has received from distinguished educators in all parts of the country. He also desires to acknowledge his obligation to Mr. Charles Hamilton, the Superintendent of the composition room of the Athenæum Press, and to Mr. I. F. White, the compositor, for the excellent typography of the book.

Criticisms and corrections will be thankfully received.

G. A. WENTWORTH.

EXETER, N. H., June, 1899.

ΒΙΒΛΙΟΘΗΚΗ

NOTE TO TEACHERS.

It is intended to have the first fourteen pages of this book simply read in the class, with such running comment and discussion as may be useful to help the beginner catch the spirit of the subject-matter, and not leave him to the mere letter of dry definitions. In like manner, the definitions at the beginning of each Book should be read and discussed in the recitation room. There is a decided advantage in having the definitions for each Book in a single group so that they can be included in one survey and discussion.

For a similar reason the theorems of limits are considered together. The subject of limits is exceedingly interesting in itself, and it was thought best to include in the theory of limits in the second Book every principle required for Plane and Solid Geometry.

When the pupil is reading each Book for the first time, it will be well to let him write his proofs on the blackboard in his own language, care being taken that his language be the simplest possible, that the arrangement of work be vertical, and that the figures be accurately constructed.

This method will furnish a valuable exercise as a language lesson, will cultivate the habit of neat and orderly arrangement of work, and will allow a brief interval for deliberating on each step.

After a Book has been read in this way, the pupil should review the Book, and should be required to draw the figures free-hand. He should state and prove the propositions orally, using a pointer to indicate on the figure every line and angle named. He should be encouraged, in reviewing each Book, to do the original exercises; to state the converse propositions, and determine whether they are true or false; and also to give well-considered answers to questions which may be asked him on many propositions.

The Teacher is strongly advised to illustrate, geometrically and arithmetically, the principles of limits. Thus, a rectangle with a constant base b, and a variable altitude x, will afford an obvious illustration of the truth that the product of a constant and a variable is also a variable; and that the limit of the product of a constant and a variable is the product of the constant by the limit of the variable. If x increases and approaches the altitude a as a limit, the area of the rectangle increases and approaches the area of the rectangle ab as a limit; if, however, x decreases and approaches zero as a limit, the area of the rectangle decreases and approaches zero as a limit.

An arithmetical illustration of this truth may be given by multiplying the approximate values of any repetend by a constant. If, for example, we take the repetend 0.3333 etc., the approximate values of the repetend will be $\frac{3}{10}$, $\frac{33}{100}$, $\frac{333}{1000}$, $\frac{3333}{10000}$, etc., and these values multiplied by 60 give the series 18, 19.8, 19.98, 19.998, etc., which evidently approaches 20 as a limit; but the product of 60 into $\frac{1}{3}$ (the limit of the repetend 0.333 etc.) is also 20.

Again, if we multiply 60 into the different values of the decreasing series $\frac{1}{30}$, $\frac{1}{300}$, $\frac{1}{3000}$, $\frac{1}{30000}$, etc., which approaches zero as a limit, we shall get the decreasing series 2, $\frac{1}{5}$, $\frac{1}{50}$, $\frac{1}{500}$, etc.; and this series evidently approaches zero as a limit.

The Teacher is likewise advised to give frequent written examinations. These should not be too difficult, and sufficient time should be allowed for accurately constructing the figures, for choosing the best language, and for determining the best arrangement.

The time necessary for the reading of examination books will be diminished by more than one half, if the use of symbols is allowed.

EXETER, N. H., 1899.

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GEOMETRY.

INTRODUCTION.

1. If a block of wood or stone is cut in the shape represented in Fig. 1, it will have *six flat faces*.

Each face of the block is called a **surface**; and if the faces are made smooth by polishing, so that, when a straight edge is applied to any one of them, the straight edge in every part will touch the surface, the faces are called **plane surfaces**, or **planes**.

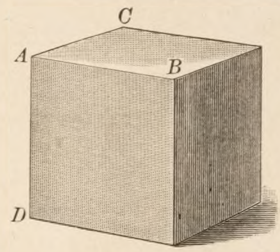


FIG. 1.

2. The intersection of any two of these surfaces is called a **line**.

3. The intersection of any three of these lines is called a **point**.

4. The block extends in three principal directions:

From left to right, A to B.
From front to back, A to C.
From top to bottom, A to D.

These are called the **dimensions** of the block, and are named in the order given, **length**, **breadth** (or *width*), and **thickness** (*height* or *depth*).

5. A **solid**, in common language, is a limited portion of space *filled with matter;* but in Geometry we have nothing to do with *the matter* of which a body is composed; we study simply its *shape* and *size;* that is, we regard a solid as a limited portion of space which may be occupied by a physical body, or marked out in some other way. Hence,

A geometrical solid is a limited portion of space.

6. The surface of a solid is simply the boundary of the solid, that which separates it from surrounding space. The surface is no part of a solid and has no thickness. Hence,

A surface has only two dimensions, length and breadth.

7. A line is simply a boundary of a surface, or the intersection of two surfaces. Since the surfaces have no thickness, a line has no thickness. Moreover, a line is no part of a surface and has no width. Hence,

A line has only one dimension, length.

8. A point is simply the extremity of a line, or the intersection of two lines. A point, therefore, has no thickness, width, or length; therefore, no magnitude. Hence,

A point has no dimension, but denotes position simply.

9. It must be distinctly understood at the outset that the points, lines, surfaces, and solids of Geometry are *purely ideal,* though they are represented to the eye in a material way. Lines, for example, drawn on paper or on the blackboard, will have some width and some thickness, and will so far fail of being *true lines;* yet, when they are used to help the mind in reasoning, it is assumed that they represent true lines, without breadth and without thickness.

10. A point is *represented* to the eye by a fine dot, and named by a letter, as A (Fig. 2). A line is named by two letters, placed one at each end, as BF. A surface is represented and named by the lines which bound it, as $BCDF$. A solid is represented by the faces which bound it.

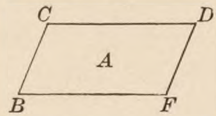


FIG. 2.

11. A point in space may be considered by itself, without reference to a line.

12. If a point moves in space, its path is a line. This line may be considered apart from the idea of a surface.

13. If a line moves in space, it generates, in general, a surface. A surface can then be considered apart from the idea of a solid.

14. If a surface moves in space, it generates, in general, a solid.

Thus, let the upright surface $ABCD$ (Fig. 3) move to the right to the position $EFGH$, the points A, B, C, and D generating the lines AE, BF, CG, and DH, respectively. The lines AB, BC, CD, and DA will generate the surfaces AF, BG, CH, and DE, respectively. The surface $ABCD$ will generate the solid AG.

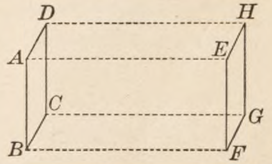


FIG. 3.

15. **Geometry** is the science which treats of *position*, *form*, and *magnitude*.

16. A **geometrical figure** is a combination of points, lines, surfaces, or solids.

17. **Plane Geometry** treats of figures all points of which are in the same plane.

Solid Geometry treats of figures all points of which are not in the same plane.

GENERAL TERMS.

18. A **proof** is a course of reasoning by which the truth or falsity of any statement is logically established.

19. An **axiom** is a statement admitted to be true without proof.

20. A **theorem** is a statement to be proved.

21. A **construction** is the representation of a required figure by means of points and lines.

22. A **postulate** is a construction admitted to be possible.

23. A **problem** is a construction to be made so that it shall satisfy certain given conditions.

24. A **proposition** is an axiom, a theorem, a postulate, or a problem.

25. A **corollary** is a truth that is easily deduced from known truths.

26. A **scholium** is a remark upon some particular feature of a proposition.

27. The **solution of a problem** consists of four parts:

1. The *analysis*, or course of thought by which the construction of the required figure is discovered.

2. The *construction* of the figure with the aid of ruler and compasses.

3. The *proof* that the figure satisfies all the conditions.

4. The *discussion* of the limitations, if any, within which the solution is possible.

28. A theorem consists of two parts: the **hypothesis**, or that which is assumed; and the **conclusion**, or that which is asserted to follow from the hypothesis.

29. The **contradictory** of a theorem is a theorem which must be true if the given theorem is false, and must be false if the given theorem is true. Thus,

A theorem: If A is B, then C is D.
Its contradictory: If A is B, then C is not D.

30. The **opposite** of a theorem is obtained by making *both the hypothesis and the conclusion negative.* Thus,

A theorem: If A is B, then C is D.
Its opposite: If A is not B, then C is not D.

31. The **converse** of a theorem is obtained by *interchanging the hypothesis and conclusion.* Thus,

A theorem: If A is B, then C is D.
Its converse: If C is D, then A is B.

32. The converse of a truth is not *necessarily* true.

Thus, Every horse is a quadruped is true, but the converse, Every quadruped is a horse, is not true.

33. *If a direct proposition and its opposite are true, the converse proposition is true; and if a direct proposition and its converse are true, the opposite proposition is true.*

Thus, if it were true that

1. If an animal is a horse, the animal is a quadruped;

2. If an animal is not a horse, the animal is not a quadruped;

it would follow that

3. If an animal is a quadruped, the animal is a horse.

Moreover, if 1 and 3 were true, then 2 would be true.

34. GENERAL AXIOMS.

1. Magnitudes which are equal to the same magnitude, or equal magnitudes, are equal to each other.

2. If equals are added to equals, the sums are equal.

3. If equals are taken from equals, the remainders are equal.

4. If equals are added to unequals, the sums are unequal in the same order; if unequals are added to unequals in the same order, the sums are unequal in that order.

5. If equals are taken from unequals, the remainders are unequal in the same order; if unequals are taken from equals, the remainders are unequal in the reverse order.

6. The doubles of the same magnitude, or of equal magnitudes, are equal; and the doubles of unequals are unequal.

7. The halves of the same magnitude, or of equal magnitudes, are equal; and the halves of unequals are unequal.

8. The whole is greater than any of its parts.

9. The whole is equal to the sum of all its parts.

35. SYMBOLS AND ABBREVIATIONS.

$>$ is (or are) greater than.	Def. . . . definition.
$<$ is (or are) less than.	Ax. . . . axiom.
$\backsimeq$ is (or are) equivalent to.	Hyp. . . hypothesis.
$\therefore$ therefore.	Cor. . . . corollary.
$\perp$ perpendicular.	Scho. . . scholium.
$\perp_s$ perpendiculars.	Ex. . . . exercise.
$\parallel$ parallel. $\parallel_s$ parallels.	Adj. . . . adjacent.
$\angle$ angle. $\angle_s$ angles.	Iden. . . identical.
$\triangle$ triangle. $\triangle_s$ triangles.	Const. . . construction.
$\square$ parallelogram.	Sup. . . . supplementary.
$\square_s$ parallelograms.	Ext. . . . exterior.
$\odot$ circle. $\odot_s$ circles.	Int. . . . interior.
rt. right. st. straight.	Alt. . . . alternate.

Q.E.D. stands for quod erat demonstrandum, *which was to be proved.*

Q.E.F. stands for quod erat faciendum, *which was to be done.*

The signs $+$, $-$, $\times$, $\div$, $=$, have the same meaning as in Algebra.

The sign + means that the Problem is found very difficult.
" — " " " is not understood.
" • " " " is solved.
(as far as progress)

PLANE GEOMETRY.

Book I.

RECTILINEAR FIGURES.

DEFINITIONS.

36. A **straight line** is a line such that any part of it, however placed on any other part, will lie wholly in that part if its extremities lie in that part, as *AB*.

37. A **curved line** is a line no part of which is straight, as *CD*.

38. A **broken line** is made up of different straight lines, as *EF*.

Fig. 4.

Note. A straight line is often called simply a *line*.

39. A **plane surface**, or a **plane**, is a surface in which, if any two points are taken, the straight line joining these points lies wholly in the surface.

40. A **curved surface** is a surface no part of which is plane.

41. A **plane figure** is a figure all points of which are in the same plane.

42. Plane figures which are bounded by straight lines are called **rectilinear** figures; by curved lines, **curvilinear** figures.

43. Figures that have the *same shape* are called **similar.** Figures that have the *same size but not the same shape* are called **equivalent.** Figures that have the *same shape and the same size* are called **equal** or **congruent.**

THE STRAIGHT LINE.

44. Postulate. A straight line can be drawn from one point to another.

45. Postulate. A straight line can be produced indefinitely.

46. Axiom.* *Only one straight line can be drawn from one point to another.* Hence, two points *determine* a straight line.

47. Cor. 1. *Two straight lines which have two points in common coincide and form but one line.*

48. Cor. 2. *Two straight lines can intersect in only one point.*

For if they had two points common, they would coincide and not intersect.

Hence, two intersecting lines *determine* a point.

49. Axiom. *A straight line is the shortest line that can be drawn from one point to another.*

50. Def. The **distance** between two points is the length of the straight line that joins them.

51. A straight line determined by two points may be considered as prolonged indefinitely.

52. If only the part of the line between two fixed points is considered, this part is called a **segment** of the line.

53. For brevity, we say "the line AB," to designate a segment of a line limited by the points A and B.

54. If a line is considered as extending from a fixed point, this point is called the **origin** of the line.

* The general axioms on page 6 apply to all magnitudes. Special geometrical axioms will be given when required.

55. If any point, C, is taken in a given straight line, AB, the two parts CA and CB are said to have *opposite directions* from the point C (Fig. 5).

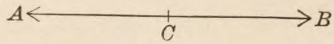


FIG. 5.

Every straight line, as AB, may be considered as extending in either of two opposite directions, namely, from A towards B, which is expressed by AB, and read segment AB; and from B towards A, which is expressed by BA, and read segment BA.

56. If the magnitude of a given line is changed, it becomes longer or shorter.

Thus (Fig. 5), by prolonging AC to B we add CB to AC, and $AB = AC + CB$. By diminishing AB to C, we subtract CB from AB, and $AC = AB - CB$.

If a given line increases so that it is prolonged by its own magnitude several times in succession, the line is *multiplied*, and the resulting line is called a *multiple* of the given line.

Thus (Fig. 6), if $AB = BC = CD = DE$, then $AC = 2\,AB$, $AD = 3\,AB$, and $AE = 4\,AB$. Hence,

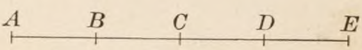


FIG. 6.

Lines of given length may be added and subtracted; they may also be multiplied by a number.

THE PLANE ANGLE.

57. The *opening* between two straight lines drawn from the same point is called a **plane angle**. The two lines, ED and EF, are called the **sides**, and E, the point of meeting, is called the **vertex** of the angle.

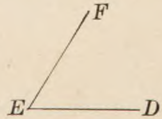


FIG. 7.

The size of an angle depends upon the *extent of opening* of its sides, and not upon the length of its sides.

58. If there is but one angle at a given vertex, the angle is designated by a capital letter placed at the vertex, and is read by simply naming the letter.

If two or more angles have the same vertex, each angle is designated by three letters, and is read by naming the three letters, the one at the vertex between the others. Thus, DAC (Fig. 8) is the angle formed by the sides AD and AC.

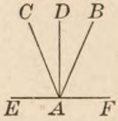


FIG. 8.

An angle is often designated by placing a small *italic* letter between the sides and near the vertex, as in Fig. 9.

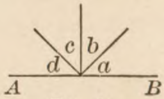


FIG. 9.

59. Postulate of Superposition. Any figure may be moved from one place to another without altering its size or shape.

60. The **test of equality** of two geometrical magnitudes is that they may be made to coincide throughout their whole extent. Thus,

Two straight lines are equal, if they can be placed one upon the other so that the points at their extremities coincide.

Two angles are equal, if they can be placed one upon the other so that their vertices coincide and their sides coincide, each with each.

61. A line or plane that divides a geometric magnitude into *two equal parts* is called the **bisector** of the magnitude.

If the angles BAD and CAD (Fig. 8) are equal, AD *bisects* the angle BAC.

62. Two angles are called **adjacent angles** when they have the same vertex and a common side between them; as the angles BOD and AOD (Fig. 10).

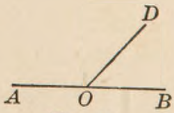


FIG. 10.

63. When one straight line meets another straight line and makes the *adjacent angles equal*, each of these angles is called a **right angle**; as angles DCA and DCB (Fig. 11).

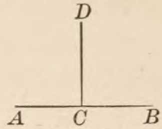


FIG. 11.

64. A **perpendicular** to a straight line is a straight line that makes a right angle with it.

Thus, if the angle DCA (Fig. 11) is a right angle, DC is perpendicular to AB, and AB is perpendicular to DC.

65. The point (as C, Fig. 11) where a perpendicular meets another line is called the **foot** of the perpendicular.

66. When the sides of an angle extend in opposite directions, so as to be in the same straight line, the angle is called a **straight angle**.

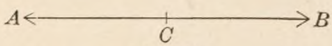


FIG. 12.

Thus, the angle formed at C (Fig. 12) with its sides CA and CB extending in opposite directions from C is a straight angle.

67. COR. *A right angle is half a straight angle.*

68. An angle less than a right angle is called an **acute angle**; as, angle A (Fig. 13).

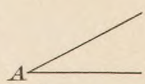


FIG. 13.

69. An angle greater than a right angle and less than a straight angle is called an **obtuse angle**; as, angle AOD (Fig. 14).

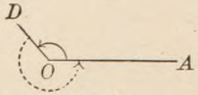


FIG. 14.

70. An angle greater than a straight angle and less than two straight angles is called a **reflex angle**; as, angle DOA, indicated by the dotted line (Fig. 14).

71. Angles that are neither right nor straight angles are called **oblique angles**; and intersecting lines that are not perpendicular to each other are called **oblique lines.**

EXTENSION OF THE MEANING OF ANGLES.

72. Suppose the straight line OC (Fig. 15) to move in the plane of the paper from coincidence with OA, about the point O as a pivot, to the position OC; then the line OC describes or generates *the angle* AOC, and the magnitude of the angle AOC depends upon the *amount of rotation* of the line from the position OA to the position OC.

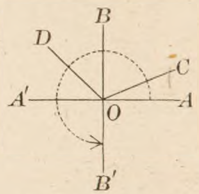


FIG. 15.

If the rotating line moves from the position OA to the position OB, *perpendicular* to OA, it generates the right angle AOB; if it moves to the position OD, it generates the obtuse angle AOD; if it moves to the position OA', it generates the straight angle AOA'; if it moves to the position OB', it generates the reflex angle AOB', indicated by the dotted line; and if it moves to the position OA again, it generates two straight angles. Hence,

73. *The angular magnitude about a point in a plane is equal to two straight angles, or four right angles; and the angular magnitude about a point on one side of a straight line drawn through the point is equal to a straight angle, or two right angles.*

74. The whole angular magnitude about a point in a plane is called a **perigon**; and two angles whose sum is a perigon are called **conjugate angles.**

NOTE. This *extension of the meaning of angles* is necessary in the applications of Geometry, as in Trigonometry, Mechanics, etc.

75. When two angles have the same vertex, and the sides of the one are prolongations of the sides of the other, they are called **vertical angles**; as, angles *a* and *b*, *c* and *d* (Fig. 16).

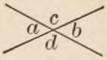


FIG. 16.

76. Two angles are called **complementary** when their sum is equal to a right angle; and each is called the *complement* of the other; as angles DOB and DOC (Fig. 17).

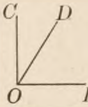


FIG. 17.

77. Two angles are called **supplementary** when their sum is equal to a straight angle; and each is called the *supplement* of the other; as, angles DOB and DOA (Fig. 18).

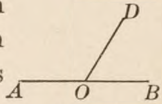


FIG. 18.

UNIT OF ANGLES.

78. By adopting a suitable unit for measuring angles we are able to express the magnitudes of angles by numbers.

If we suppose OC (Fig. 15) to turn about O from coincidence with OA until it makes *one three hundred sixtieth* of a revolution, it generates an angle at O, which is taken as the unit for measuring angles. This unit is called a *degree.*

The degree is subdivided into sixty equal parts, called *minutes*, and the minute into sixty equal parts, called *seconds.*

Degrees, minutes, and seconds are denoted by symbols. Thus, 5 degrees 13 minutes 12 seconds is written $5° \, 13' \, 12''$.

A right angle is generated when OC has made *one fourth* of a revolution and contains $90°$; a straight angle, when OC has made *half* of a revolution and contains $180°$; and a perigon, when OC has made a complete revolution and contains $360°$.

NOTE. The natural angular unit is one complete revolution. But this unit would require us to express the values of most angles by fractions. The advantage of using the degree as the unit consists in its convenient size, and in the fact that 360 is divisible by so many different integral numbers.

79. By the method of superposition we are able to compare magnitudes of the same kind. Suppose we have two angles, ABC and DEF (Fig. 19). Let the side ED be placed on the side BA, so that the vertex E shall fall on B; then, if the side EF falls on BC, the angle DEF equals the angle ABC; if the side EF falls between BC and BA in the position shown by the dotted line BG, the angle DEF is less than the angle ABC; but if the side EF falls in the position shown by the dotted line BH, the angle DEF is greater than the angle ABC.

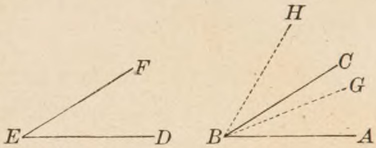


FIG. 19.

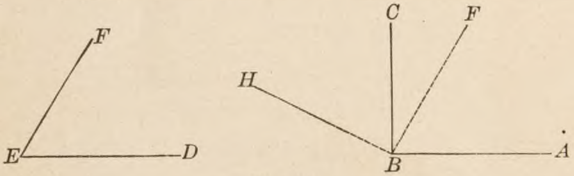


FIG. 20.

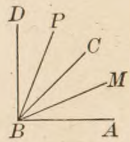


FIG. 21.

80. If we have the angles ABC and DEF (Fig. 20), and place the vertex E on B and the side ED on BC, so that the angle DEF takes the position CBH, the angles DEF and ABC will together be equal to the angle ABH.

If the vertex E is placed on B, and the side ED on BA, so that the angle DEF takes the position ABF, the angle FBC will be the difference between the angles ABC and DEF.

If an angle is increased by its own magnitude two or more times in succession, the angle is *multiplied* by a number.

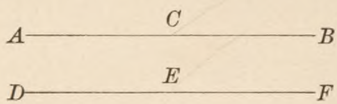
Thus, if the angles ABM, MBC, CBP, PBD (Fig. 21) are all equal, the angle ABD is 4 times the angle ABM. Therefore,

Angles may be added and subtracted; they may also be multiplied by a number.

PERPENDICULAR AND OBLIQUE LINES.

Proposition I. Theorem.

81. *All straight angles are equal.*



Let the angles ACB and DEF be any two straight angles.

To prove that $\angle ACB = \angle DEF$.

Proof. Place the $\angle ACB$ on the $\angle DEF$, so that the vertex C shall fall on the vertex E, and the side CB on the side EF.

Then CA will fall on ED, § 47

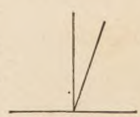
(*because ACB and DEF are straight lines*).

$\therefore \angle ACB = \angle DEF$. § 60

Q. E. D.

82. Cor. 1. *All right angles are equal.* Ax. 7

83. Cor. 2. *At a given point in a given line there can be but one perpendicular to the line.*



For, if there could be two ⊥s, we should have rt. $\angle$s of different magnitudes; but this is impossible, § 82.

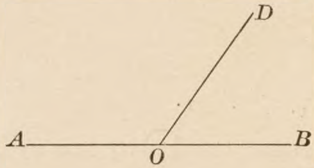
84. Cor. 3. *The complements of the same angle or of equal angles are equal.* Ax. 3

85. Cor. 4. *The supplements of the same angle or of equal angles are equal.* Ax. 3

Note. The beginner must not forget that in Plane Geometry all the points of a figure are in the same plane. Without this restriction in Cor. 2, an indefinite number of perpendiculars can be erected at a given point in a given line.

PROPOSITION II. THEOREM.

86. *If two adjacent angles have their exterior sides in a straight line, these angles are supplementary.*



Let the exterior sides OA and OB of the adjacent angles AOD and BOD be in the straight line AB.

To prove that $\measuredangle s$ *AOD and BOD are supplementary.*

Proof. AOB is a straight line. Hyp.

$\therefore \angle AOB$ is a st. $\angle$. § 66

But $\angle AOD + \angle BOD =$ the st. $\angle AOB$. Ax. 9

$\therefore$ the $\measuredangle s$ AOD and BOD are supplementary. § 77

Q.E.D.

87. DEF. Adjacent angles that are supplements of each other are called *supplementary-adjacent angles.*

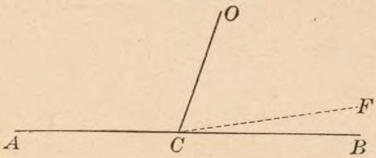
Since the angular magnitude about a point is neither increased nor diminished by the number of lines which radiate from the point, it follows that,

88. COR. 1. *The sum of all the angles about a point in a plane is equal to a perigon, or two straight angles.*

89. COR. 2. *The sum of all the angles about a point in a plane, on the same side of a straight line passing through the point, is equal to a straight angle, or two right angles.*

Proposition III. Theorem.

90. Conversely: *If two adjacent angles are supplementary, their exterior sides are in the same straight line.*



Let the adjacent angles OCA and OCB be supplementary.

To prove that AC and CB are in the same straight line.

Proof. Suppose CF to be in the same line with AC.

Then $\angle s$ OCA and OCF are supplementary, § 86

(*if two adjacent angles have their exterior sides in a straight line, these angles are supplementary*).

But $\angle s$ OCA and OCB are supplementary. Hyp.

$\therefore$ $\angle s$ OCF and OCB have the same supplement.

$\therefore \angle OCF = \angle OCB$. § 85

$\therefore$ CB and CF coincide. § 60

$\therefore$ AC and CB are in the same straight line. Q. E. D.

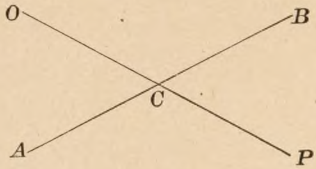
Since Propositions II. and III. are true, their opposites are true. Hence, § 33

91. Cor. 1. *If the exterior sides of two adjacent angles are not in a straight line, these angles are not supplementary.*

92. Cor. 2. *If two adjacent angles are not supplementary, their exterior sides are not in the same straight line.*

Proposition IV. Theorem.

93. *If one straight line intersects another straight line, the vertical angles are equal.*



Let the lines OP and AB intersect at C.

To prove that $\angle OCB = \angle ACP$.

Proof. $\angle OCA$ and $\angle OCB$ are supplementary. § 86

$\angle OCA$ and $\angle ACP$ are supplementary, § 86

(*if two adjacent angles have their exterior sides in a straight line, these angles are supplementary*).

$\therefore$ $\angle$s OCB and ACP have the same supplement.

$\therefore \angle OCB = \angle ACP$. § 85

In like manner, $\angle ACO = \angle PCB$.

Q.E.D.

94. Cor. *If one of the four angles formed by the intersection of two straight lines is a right angle, the other three angles are right angles.*

Compl = 41°
Suppl = 131°

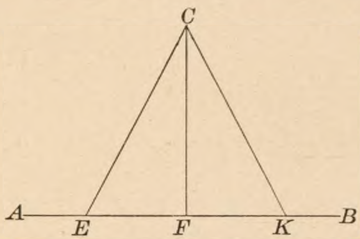
Ex. 1. Find the complement and the supplement of an angle of 49°.

Ex. 2. Find the number of degrees in an angle if it is double its complement; if it is one fourth of its complement.

Ex. 3. Find the number of degrees in an angle if it is double its supplement; if it is one third of its supplement.

PROPOSITION V. THEOREM.

95. *Two straight lines drawn from a point in a perpendicular to a given line, cutting off on the given line equal segments from the foot of the perpendicular, are equal and make equal angles with the perpendicular.*



Let CF be a perpendicular to the line AB, and CE and CK two straight lines cutting off on AB equal segments FE and FK from F.

To prove that $CE = CK$; *and* $\angle FCE = \angle FCK$.

Proof. Fold over CFA, on CF as an axis, until it falls on the plane at the right of CF.

FA will fall along FB,

(*since* $\angle CFA = \angle CFB$, *each being a rt.* $\angle$, *by hyp.*).

Point E will fall on point K,

(*since* $FE = FK$, *by hyp.*).

$\therefore CE = CK$, § 60

(*their extremities being the same points*);

and $\angle FCE = \angle FCK$, § 60

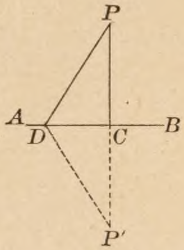
(*since their vertices coincide, and their sides coincide, each with each*).

Q. E. D.

Ex. 4. Find the number of degrees in the angle included by the hands of a clock at 1 o'clock. 3 o'clock. 5 o'clock. 6 o'clock.

PROPOSITION VI. THEOREM.

96. *Only one perpendicular can be drawn to a given line from a given external point.*



Let AB be the given line, P the given external point, PC a perpendicular to AB from P, and PD any other line from P to AB.

To prove that PD *is not* $\perp$ *to* AB.

Proof. Produce PC to P', making CP' equal to PC.

Draw DP'.

By construction, PCP' is a straight line.

$\therefore PDP'$ is not a straight line, § 46

(*only one straight line can be drawn from one point to another*).

Hence, $\angle PDP'$ is not a straight angle.

Since PC is $\perp$ to DC, and $PC = CP'$,

AC is $\perp$ to PP' at its middle point.

$\therefore \angle PDC = \angle P'DC$, § 95

(*two straight lines from a point in a $\perp$ to a line, cutting off on the line equal segments from the foot of the $\perp$, make equal $\angle$s with the $\perp$*).

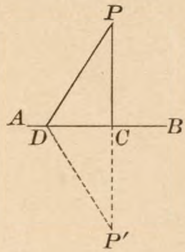
Since $\angle PDP'$ is not a straight angle,

$\angle PDC$, the half of $\angle PDP'$, is not a right angle.

$\therefore PD$ is not $\perp$ to AB. Q. E. D.

Proposition VII. Theorem.

97. *The perpendicular is the shortest line that can be drawn to a straight line from an external point.*



Let AB be the given straight line, P the given point, PC the perpendicular, and PD any other line drawn from P to AB.

To prove that $PC < PD.$

Proof. Produce PC to P', making $CP' = PC$.

Draw DP'.

Then $PD = DP'$, § 95

(*two straight lines drawn from a point in a* $\perp$ *to a line, cutting off on the line equal segments from the foot of the* $\perp$, *are equal*).

$$\therefore PD + DP' = 2\,PD,$$

and $PC + CP' = 2\,PC.$ Const.

But $PC + CP' < PD + DP'.$ § 49

$$\therefore 2\,PC < 2\,PD.$$

$$\therefore PC < PD.$$ Ax. 7

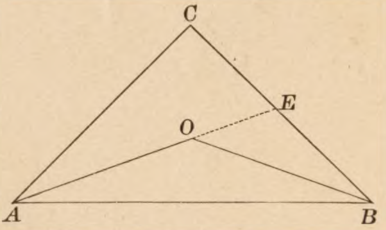
Q. E. D.

98. Cor. *The shortest line that can be drawn from a point to a given line is perpendicular to the given line.*

99. Def. The **distance** of a point from a line is the length of the perpendicular from the point to the line.

PROPOSITION VIII. THEOREM.

100. *The sum of two lines drawn from a point to the extremities of a straight line is greater than the sum of two other lines similarly drawn, but included by them.*



Let CA and CB be two lines drawn from the point C to the extremities of the straight line AB. Let OA and OB be two lines similarly drawn, but included by CA and CB.

To prove that $CA + CB > OA + OB.$

Proof. Produce AO to meet the line CB at E.

Then $CA + CE > OA + OE,$

and $BE + OE > OB,$ § 49

(*a straight line is the shortest line from one point to another*).

Add these inequalities, and we have

$CA + CE + BE + OE > OA + OE + OB.$ Ax. 4

Substitute for $CE + BE$ its equal CB, then

$CA + CB + OE > OA + OE + OB.$

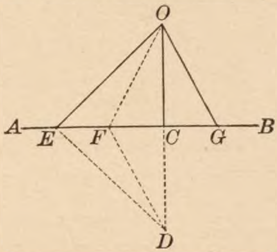
Take away OE from each side of the inequality.

$CA + CB > OA + OB.$ Ax. 5

Q. E. D.

Proposition IX. Theorem.

101. *Of two straight lines drawn from the same point in a perpendicular to a given line, cutting off on the line unequal segments from the foot of the perpendicular, the more remote is the greater.*



Let OC be perpendicular to AB, OG and OE two straight lines to AB, and CE greater than CG.

To prove that $OE > OG$.

Proof. Take CF equal to CG, and draw OF.

Then $OF = OG$, § 95

(*two straight lines drawn from a point in a* $\perp$ *to a line, cutting off on the line equal segments from the foot of the* $\perp$, *are equal*).

Produce OC to D, making $CD = OC$.

Draw ED and FD.

Then $OE = ED$, and $OF = FD$. § 95

But $OE + ED > OF + FD$, § 100

$\therefore 2\,OE > 2\,OF$, $OE > OF$, and $OE > OG$. Q.E.D.

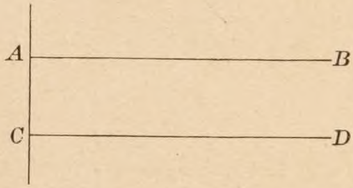
102. Cor. *Only two equal straight lines can be drawn from a point to a straight line; and of two unequal lines, the greater cuts off on the line the greater segment from the foot of the perpendicular.*

PARALLEL LINES.

103. Def. Two **parallel lines** are lines that lie in the same plane and cannot meet however far they are produced.

Proposition X. Theorem.

104. *Two straight lines in the same plane perpendicular to the same straight line are parallel.*



Let AB and CD be perpendicular to AC.

To prove that AB and CD are parallel.

Proof. If AB and CD are not parallel, they will meet if sufficiently prolonged, and we shall have two perpendicular lines from their point of meeting to the same straight line; but this is impossible, § 96

(*only one perpendicular can be drawn to a given line from a given external point*).

$\therefore$ AB and CD are parallel. Q. E. D.

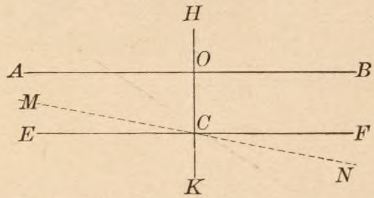
105. Axiom. *Through a given point only one straight line can be drawn parallel to a given straight line.*

106. Cor. *Two straight lines in the same plane parallel to a third straight line are parallel to each other.*

For if they could meet, we should have two straight lines from the point of meeting parallel to a straight line; but this is impossible. § 105

Proposition XI. Theorem.

107. *If a straight line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*



Let AB and EF be two parallel lines, and let HK be perpendicular to AB, and cut EF at C.

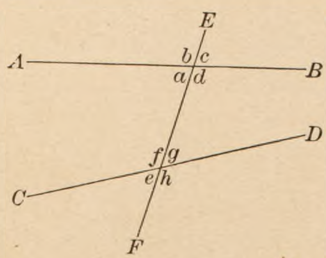
To prove that HK *is* $\perp$ *to* EF.

Proof. Suppose MN drawn through $C \perp$ to HK.

Then	MN is $\parallel$ to AB.	§ 104
But	EF is $\parallel$ to AB.	Hyp.
	$\therefore EF$ coincides with MN.	§ 105
But	MN is $\perp$ to HK.	Const.
	$\therefore EF$ is $\perp$ to HK,	
that is,	HK is $\perp$ to EF.	Q. E. D.

108. Def. A straight line that cuts two or more straight lines is called a **transversal** of those lines.

109. If the transversal EF cuts AB and CD, the angles a, d, g, f are called *interior* angles; b, c, h, e are called *exterior* angles.

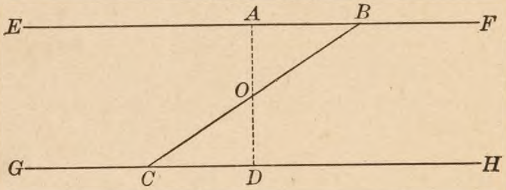


The angles d and f, and a and g, are called *alternate-interior* angles; the angles b and h, and c and e, are called *alternate-exterior* angles.

The angles b and f, c and g, e and a, h and d, are called *exterior-interior* angles.

PROPOSITION XII. THEOREM.

110. *If two parallel lines are cut by a transversal, the alternate-interior angles are equal.*



Let EF and GH be two parallel lines cut by the transversal BC.

To prove that $\angle EBC = \angle BCH.$

Proof. Through O, the middle point of BC, suppose AD drawn $\perp$ to GH.

Then AD is likewise $\perp$ to EF, § 107

(*a straight line* $\perp$ *to one of two* $\parallel$s *is* $\perp$ *to the other*),

that is, CD and BA are both $\perp$ to AD.

Apply the figure COD to the figure BOA, so that OD shall fall along OA.

Then OC will fall along OB, § 93

(*since* $\angle COD = \angle BOA$, *being vertical* $\angle$s);

and C will fall on B,

(*since* $OC = OB$, *by construction*).

Then the $\perp$ CD will fall along the $\perp$ BA, § 96

(*only one* $\perp$ *can be drawn to a given line from a given external point*).

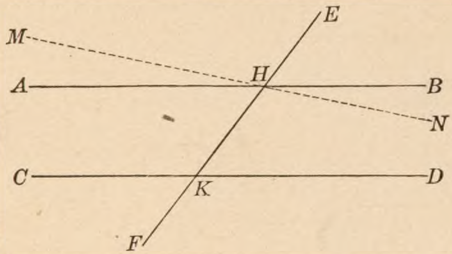
$\therefore$ $\angle OCD$ coincides with $\angle OBA$, and is equal to it, § 60

(*two angles are equal, if their vertices coincide and their sides coincide, each with each*).

Q. E. D.

Proposition XIII. Theorem.

111. Conversely: *When two straight lines in the same plane are cut by a transversal, if the alternate-interior angles are equal, the two straight lines are parallel.*



ΒΙΒΛΙΟΘΗΚΗ
Μ. ΟΛΛΑΤΑΠΣ

Let EF cut the straight lines AB and CD in the points H and K, and let the angles AHK and HKD be equal.

To prove that AB *is* $\parallel$ *to* CD.

Proof. Suppose MN drawn through $H \parallel$ to CD.

Then $\angle MHK = \angle HKD$, § 110

(*being alt.-int.* $\angle$s *of* $\parallel$ *lines*).

But $\angle AHK = \angle HKD$. Hyp.

$\therefore \angle MHK = \angle AHK$. Ax. 1

$\therefore MN$ and AB coincide. § 60

But MN is $\parallel$ to CD. Const.

$\therefore AB$, which coincides with MN, is $\parallel$ to CD.

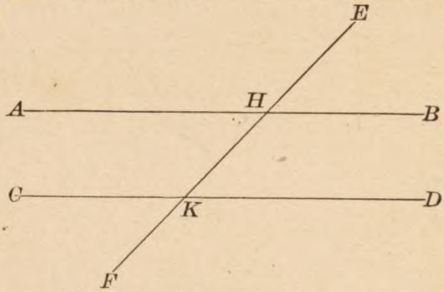
Q.E.D.

Ex. 5. Find the complement and the supplement of an angle that contains 37° 53′ 49″.

Ex. 6. If the complement of an angle is one third of its supplement, how many degrees does the angle contain ?

PROPOSITION XIV. THEOREM.

112. *If two parallel lines are cut by a transversal, the exterior-interior angles are equal.*



Let AB and CD be two parallel lines cut by the transversal EF, in the points H and K.

To prove that $\angle EHB = \angle HKD$.

Proof. $\angle EHB = \angle AHK$, § 93

(*being vertical* $\measuredangle$s).

$\angle AHK = \angle HKD$, § 110

(*being alt.-int.* $\measuredangle$s *of* $\parallel$ *lines*).

$\therefore \angle EHB = \angle HKD$. Ax. 1

In like manner $\angle EHA = \angle HKC$. Q. E. D.

113. COR. *The alternate-exterior angles* EHB *and* CKF, *and also* AHE *and* DKF, *are equal.*

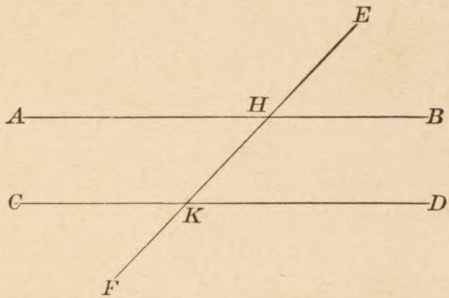
PROPOSITION XV. THEOREM.

114. CONVERSELY: *When two straight lines in a plane are cut by a transversal, if the exterior-interior angles are equal, these two straight lines are parallel.*

(Proof similar to that in § 111.)

PROPOSITION XVI. THEOREM.

115. *If two parallel lines are cut by a transversal, the two interior angles on the same side of the transversal are supplementary.*



Let AB and CD be two parallel lines cut by the transversal EF in the points H and K.

To prove that $\measuredangle s$ *BHK and HKD are supplementary.*

Proof. $\angle EHB + \angle BHK =$ a st. $\angle$, § 86

(*being sup.-adj.* $\measuredangle s$).

But $\angle EHB = \angle HKD$, § 112

(*being ext.-int.* $\measuredangle s$ *of* $\parallel$ *lines*).

$\therefore \angle BHK + \angle HKD =$ a st. $\angle$.

$\therefore \measuredangle s$ BHK and HKD are supplementary. § 77

Q. E. D.

PROPOSITION XVII. THEOREM.

116. CONVERSELY: *When two straight lines in a plane are cut by a transversal, if two interior angles on the same side of the transversal are supplementary, the two straight lines are parallel.*

(Proof similar to that in § 111.)

TRIANGLES.

117. A **triangle** is a portion of a plane bounded by three straight lines; as, ABC (Fig. 1).

The bounding lines are called the **sides** of the triangle, and their sum is called its **perimeter**; the angles included by the sides are called the **angles** of the triangle, and the vertices of these angles, the **vertices** of the triangle.

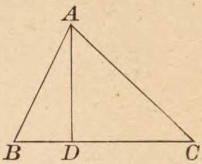


Fig. 1.

118. **Adjacent angles** of a rectilinear figure are two angles that have one side of the figure common; as, angles A and B (Fig. 2).

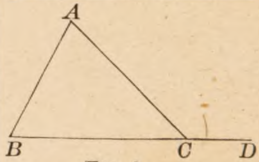
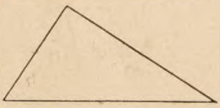
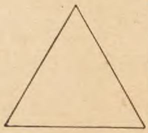


Fig. 2.

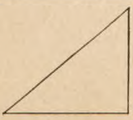
119. An **exterior angle** of a triangle is an angle included by one side and another side produced; as, ACD (Fig. 2). The interior angle ACB is adjacent to the exterior angle; the interior angles, A and B, are called **opposite interior angles.**

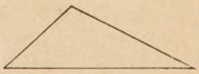

Scalene.


Isosceles.

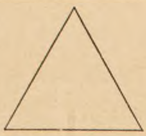

Equilateral.

120. A triangle is called a **scalene triangle** when no two of its sides are equal; an **isosceles triangle**, when two of its sides are equal; an **equilateral triangle**, when its three sides are equal.


Right.


Obtuse.


Acute.


Equiangular.

121. A triangle is called a **right triangle,** when one of its angles is a right angle; an **obtuse triangle,** when one of its angles is an obtuse angle; an **acute triangle,** when all three of its angles are acute angles; an **equiangular triangle.** when its three angles are equal.

122. In a right triangle, the side opposite the right angle is called the **hypotenuse,** and the other two sides the **legs.**

123. The side on which a triangle is supposed to stand is called the **base** of the triangle. In the isosceles triangle, the equal sides are called the legs, and the other side, the base; in other triangles, any one of the sides may be taken as the base.

124. The angle opposite the base of a triangle is called the **vertical angle,** and its vertex, the **vertex** of the triangle.

125. The **altitude** of a triangle is the perpendicular from the vertex to the base, or to the base produced; as, AD (Fig. 1).

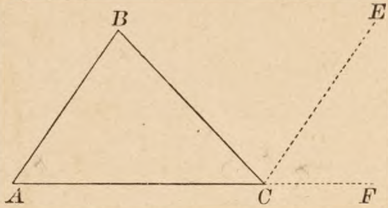
126. The three perpendiculars from the vertices of a triangle to the opposite sides (produced if necessary) are called the **altitudes** of the triangle; the three bisectors of the angles are called the **bisectors** of the triangle; and the three lines from the vertices to the middle points of the opposite sides are called the **medians** of the triangle.

127. If two triangles have the angles of the one equal, respectively, to the angles of the other, the equal angles are called **homologous angles,** and the sides opposite the equal angles are called **homologous sides.**

128. Two triangles are equal in all respects if they can be made to coincide (§ 60). The homologous sides of *equal triangles* are equal, and the homologous angles are equal.

PROPOSITION XVIII. THEOREM.

129. *The sum of the three angles of a triangle is equal to two right angles.*



Let A, B, and BCA be the angles of the triangle ABC.

To prove that $\angle A + \angle B + \angle BCA = 2$ *rt.* $\angle$s.

Proof. Suppose CE drawn $\parallel$ to AB, and prolong AC to F.

Then $\angle ECF + \angle ECB + \angle BCA = 2$ rt. $\angle$s, § 89

(*the sum of all the* $\angle$s *about a point on the same side of a straight line passing through the point is equal to 2 rt.* $\angle$s).

But $\angle A = \angle ECF$, § 112

(*being ext.-int.* $\angle$s *of the* $\parallel$ *lines* AB *and* CE),

and $\angle B = \angle BCE$, § 110

(*being alt.-int.* $\angle$s *of the* $\parallel$ *lines* AB *and* CE).

Put for the $\angle$s ECF and BCE their equals, the $\angle$s A and B.

Then $\angle A + \angle B + \angle BCA = 2$ rt. $\angle$s. Q.E.D.

130. COR. 1. *The sum of two angles of a triangle is less than two right angles.*

131. COR. 2. *If the sum of two angles of a triangle is taken from two right angles, the remainder is equal to the third angle.*

132. COR. 3. *If two triangles have two angles of the one equal to two angles of the other, the third angles are equal.*

133. Cor. 4. *If two right triangles have an acute angle of the one equal to an acute angle of the other, the other acute angles are equal.*

134. Cor. 5. *In a triangle there can be but one right angle, or one obtuse angle.*

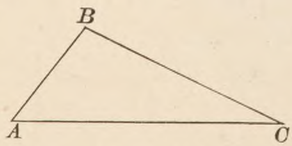
135. Cor. 6. *In a right triangle the two acute angles are together equal to one right angle, or 90°.*

136. Cor. 7. *In an equiangular triangle, each angle is one third of two right angles, or 60°.*

137. Cor. 8. *An exterior angle of a triangle is equal to the sum of the two opposite interior angles, and therefore greater than either of them.*

Proposition XIX. Theorem.

138. *The sum of two sides of a triangle is greater than the third side, and their difference is less than the third side.*



In the triangle ABC, let AC be the longest side.

To prove that $AB + BC > AC$, *and* $AC - BC < AB$.

Proof. $AB + BC > AC$, § 49

(*a straight line is the shortest line from one point to another*).

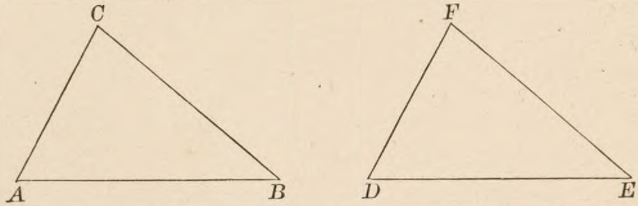
Take away BC from both sides.

Then $AB > AC - BC$, Ax. 5

or $AC - BC < AB$. Q. E. D.

PROPOSITION XX. THEOREM.

139. *Two triangles are equal if two angles and the included side of the one are equal, respectively, to two angles and the included side of the other.*



In the triangles ABC, DEF, let the angle A be equal to the angle D, B to E, and the side AB to DE.

To prove that $\triangle ABC = \triangle DEF$.

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$ so that AB shall coincide with its equal, DE.

Then AC will fall along DF, and BC along EF,

(*for* $\angle A = \angle D$, *and* $\angle B = \angle E$, *by hyp.*).

$\therefore$ C will fall on F, § 48

(*two straight lines can intersect in only one point*).

$\therefore$ the two $\triangle$s coincide, and are equal. § 60

Q. E. D.

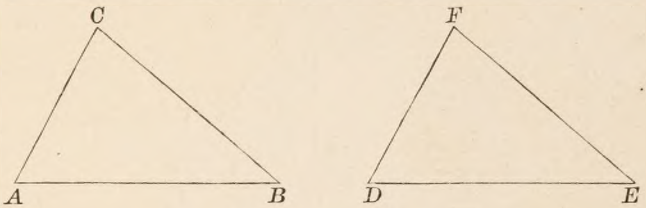
140. COR. 1. *Two triangles are equal if a side and any two angles of the one are equal to the homologous side and two angles of the other.* § 132

141. COR. 2. *Two right triangles are equal if the hypotenuse and an acute angle of the one are equal, respectively, to the hypotenuse and an acute angle of the other.* § 133

142. COR. 3. *Two right triangles are equal if a leg and an acute angle of the one are equal, respectively, to a leg and the homologous acute angle of the other.* § 133

PROPOSITION XXI. THEOREM.

143. *Two triangles are equal if two sides and the included angle of the one are equal, respectively, to two sides and the included angle of the other.*



In the triangles ABC and DEF, let AB be equal to DE, AC to DF, and the angle A to the angle D.

To prove that $\triangle ABC = \triangle DEF$.

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$ so that AB shall coincide with its equal, DE.

Then AC will fall along DF,

(*for* $\angle A = \angle D$, *by hyp.*);

and C will fall on F,

(*for* $AC = DF$, *by hyp.*).

$\therefore CB = FE$,

(*their extremities being the same points*).

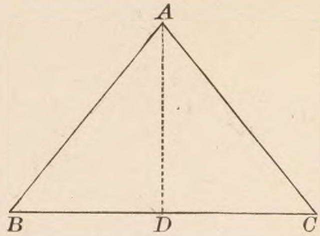
$\therefore$ the two $\triangle$s coincide, and are equal. Q. E. D.

144. COR. *Two right triangles are equal if their legs are equal, each to each.*

NOTE. In § 139 we have given two angles and the included side, in § 143 two sides and the included angle; hence, by interchanging the words *sides* and *angles*, either theorem is changed to the other. This is called the *Principle of Duality*, or the *Principle of Reciprocity*. The reciprocal of a theorem is not always true, just as the converse of a theorem is not always true.

PROPOSITION XXII. THEOREM.

145. *In an isosceles triangle the angles opposite the equal sides are equal.*



Let ABC be an isosceles triangle, having AB and AC equal.

To prove that $\angle B = \angle C$.

Proof. Suppose AD drawn so as to bisect the $\angle BAC$.

In the $\triangle$ ADB and ADC,

$AB = AC$, Hyp.

$AD = AD$, Iden.

and $\angle BAD = \angle CAD$. Const.

$\therefore \triangle ADB = \triangle ADC$, § 143

(*two* $\triangle$ *are equal if two sides and the included* $\angle$ *of the one are equal, respectively, to two sides and the included* $\angle$ *of the other*).

$\therefore \angle B = \angle C$, § 128

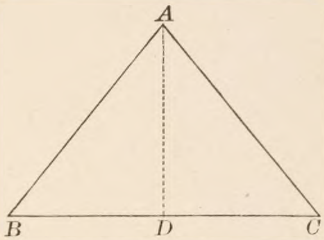
(*being homologous angles of equal triangles*). Q. E. D.

146. COR. *An equilateral triangle is equiangular, and each angle is two thirds of a right angle.*

Ex. 7. If the equal sides of an isosceles triangle are produced, the angles on the other side of the base are equal.

Proposition XXIII. Theorem.

147. *If two angles of a triangle are equal, the sides opposite the equal angles are equal, and the triangle is isosceles.*



In the triangle ABC, let the angle B be equal to the angle C.

To prove that $AB = AC$.

Proof. Suppose AD drawn $\perp$ to BC.

In the rt. $\triangle$s ADB and ADC,

$AD = AD$, Iden.

and $\angle B = \angle C$. Hyp.

$\therefore$ rt. $\triangle ADB =$ rt. $\triangle ADC$, § 142

(*having a leg and an acute $\angle$ of the one equal, respectively, to a leg and the homologous acute $\angle$ of the other*).

$\therefore AB = AC$, § 128

(*being homologous sides of equal $\triangle$s*).

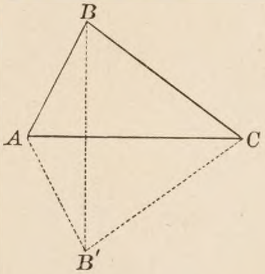
Q. E. D.

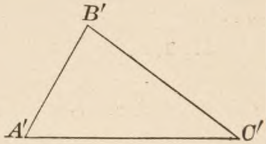
148. Cor. 1. *An equiangular triangle is also equilateral.*

149. Cor. 2. *The perpendicular from the vertex to the base of an isosceles triangle bisects the base, and bisects the vertical angle of the triangle.*

Proposition XXIV. Theorem.

150. *Two triangles are equal if the three sides of the one are equal, respectively, to the three sides of the other.*





In the triangles ABC and A′B′C′, let AB be equal to A′B′, AC to A′C′, BC to B′C′.

To prove that $\triangle ABC = \triangle A'B'C'$.

Proof. Place $\triangle A'B'C'$ in the position $AB'C$, having its greatest side $A'C'$ in coincidence with its equal AC, and its vertex at B', opposite B; and draw BB'.

Since $AB = AB'$, Hyp.

$\angle ABB' = \angle AB'B$, § 145

(in an isosceles △ the ∠s opposite the equal sides are equal).

Since $CB = CB'$, Hyp.

$\angle CBB' = \angle CB'B$. § 145

$\therefore \angle ABB' + \angle CBB' = \angle AB'B + \angle CB'B$. Ax. 2

Hence, $\angle ABC = \angle AB'C$.

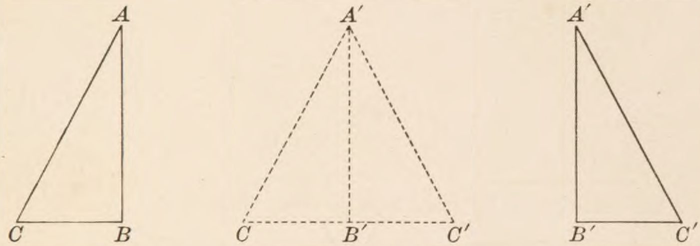
$\therefore \triangle ABC = \triangle AB'C$, § 143

(two △s are equal if two sides and the included ∠ of the one are equal, respectively, to two sides and the included ∠ of the other).

$\therefore \triangle ABC = \triangle A'B'C'$. Q.E.D.

Proposition XXV. Theorem.

151. *Two right triangles are equal if a leg and the hypotenuse of the one are equal, respectively, to a leg and the hypotenuse of the other.*



In the right triangles ABC and A′B′C′, let AB be equal to A′B′, and AC to A′C′.

To prove that $\triangle ABC = \triangle A'B'C'$.

Proof. Apply the $\triangle ABC$ to the $\triangle A'B'C'$, so that AB shall coincide with $A'B'$, A falling on A', B on B', and C and C' on opposite sides of $A'B'$.

Then BC will fall along $C'B'$ produced,

(*for* $\angle ABC = \angle A'B'C'$, *each being a rt.* $\angle$).

Since $AC = A'C'$, Hyp.

the $\triangle A'CC'$ is an isosceles triangle. § 120

$\therefore \angle C = \angle C'$, § 145

(*△ opposite the equal sides of an isosceles* $\triangle$ *are equal*).

$\therefore$ △ ABC and $A'B'C'$ are equal, § 141

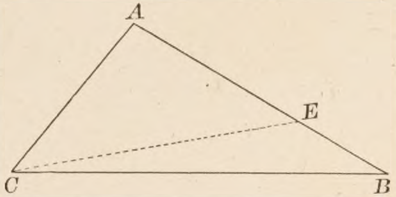
(*two right △ are equal if they have the hypotenuse and an acute* $\angle$ *of the one equal to the hypotenuse and an acute* $\angle$ *of the other*).

Q. E. D.

Ex. 8. How many degrees are there in each of the acute angles of an isosceles right triangle?

Proposition XXVI. Theorem.

152. *If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.*



In the triangle ACB, let AB be greater than AC.

To prove that $\angle ACB$ *is greater than* $\angle B$.

Proof. On AB take AE equal to AC.

Draw EC.

$\angle AEC = \angle ACE$, § 145

(*being* $\angle$s *opposite equal sides*).

But $\angle AEC$ is greater than $\angle B$, § 137

(*an exterior* $\angle$ *of a* $\triangle$ *is greater than either opposite interior* $\angle$),

and $\angle ACB$ is greater than $\angle ACE$. Ax. 8

Substitute for $\angle ACE$ its equal $\angle AEC$,

then $\angle ACB$ is greater than $\angle AEC$.

Since $\angle AEC$ is greater than $\angle B$, and $\angle ACB$ is greater than $\angle AEC$,

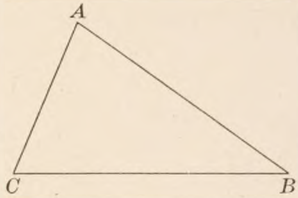
$\angle ACB$ is greater than $\angle B$. Q. E. D.

Ex. 9. If any angle of an isosceles triangle is equal to two thirds of a right angle (60°), what is the value of each of the two remaining angles?

Ex. 10. One angle of a triangle is 34°. Find the other angles, if one of them is twice the other.

PROPOSITION XXVII. THEOREM.

153. RECIPROCALLY: *If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle.*



In the triangle ACB, let the angle C be greater than the angle B.

To prove that $AB > AC$.

Proof. Now $AB = AC$, or $< AC$, or $> AC$.

But AB is not equal to AC;

for then the $\angle C$ would be equal to the $\angle B$, § 145

(*being* $\angle$s *opposite equal sides*).

And AB is not less than AC;

for then the $\angle C$ would be less than the $\angle B$. § 152

Both these conclusions are contrary to the hypothesis that the $\angle C$ is greater than the $\angle B$.

Hence, AB cannot be equal to AC or less than AC.

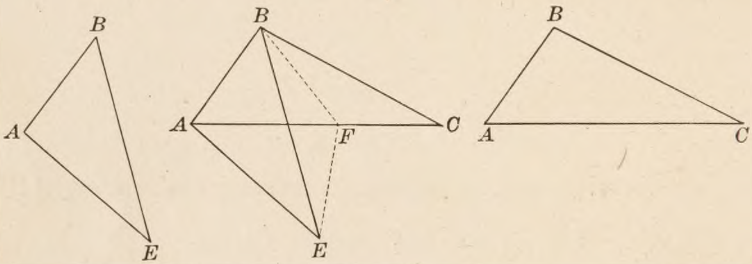
$\therefore AB > AC$. Q.E.D.

Ex. 11. If the vertical angle of an isosceles triangle is equal to 30°, find the exterior angle included by a side and the base produced.

Ex. 12. If the vertical angle of an isosceles triangle is equal to 36°, find the angle included by the bisectors of the base angles.

PROPOSITION XXVIII. THEOREM.

154. *If two triangles have two sides of the one equal, respectively, to two sides of the other, but the included angle of the first triangle greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



In the triangles ABC and ABE, let AB be equal to AB, BC to BE; but let the angle ABC be greater than the angle ABE.

To prove that $AC > AE$.

Proof. Place the $\triangle$ so that AB of the one shall fall on AB of the other, and BE within the $\angle ABC$.

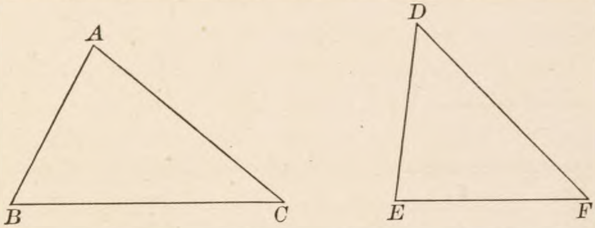
Suppose BF drawn to bisect the $\angle EBC$, and draw EF.

The $\triangle$ EBF and CBF are equal. § 143

For	$BF = BF$,	Iden.
	$BE = BC$,	Hyp.
and	$\angle EBF = \angle CBF$.	Const.
	$\therefore EF = FC$.	§ 128
Now	$AF + FE > AE$.	§ 138
	$\therefore AF + FC > AE$.	
	$\therefore AC > AE$.	Q. E. D.

PROPOSITION XXIX. THEOREM.

155. CONVERSELY: *If two sides of a triangle are equal, respectively, to two sides of another, but the third side of the first triangle is greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second.*



In the triangles ABC and DEF, let AB be equal to DE, AC to DF; but let BC be greater than EF.

To prove that the $\angle A$ is greater than the $\angle D$.

Proof. Now the $\angle A$ is equal to the $\angle D$, or less than the $\angle D$, or greater than the $\angle D$.

But the $\angle A$ is not equal to the $\angle D$;

for then the $\triangle ABC$ would be equal to the $\triangle DEF$, § 143

(having two sides and the included $\angle$ of the one equal, respectively, to two sides and the included $\angle$ of the other),

and BC would be equal to EF.

And the $\angle A$ is not less than the $\angle D$, for then BC would be less than EF. § 154

Both these conclusions are contrary to the hypothesis that BC is greater than EF.

Since the $\angle A$ is not equal to the $\angle D$ or less than the $\angle D$,

the $\angle A$ is greater than the $\angle D$. Q.E.D.

LOCI OF POINTS.

156. If it is required to find a point which shall fulfil a *single* geometric condition, the point may have an *unlimited number of positions.* If, however, all the points are in the same plane, the required point will be confined to a *particular line*, or *group of lines.*

A point in a plane at a given distance from a fixed straight line of indefinite length in that plane, is evidently in one of two straight lines, so drawn as to be everywhere at the given distance from the fixed line, one on one side of the fixed line, and the other on the other side of it.

A point in a plane equidistant from two parallel lines in that plane is evidently in a straight line drawn between the two given parallel lines and everywhere equidistant from them.

157. All points in a plane that satisfy a single geometrical condition lie, in general, in a line or group of lines; and this line or group of lines is called the **locus** of the points that satisfy the given condition.

158. To prove *completely* that a certain line is the locus of points that fulfil a given condition, it is necessary to prove

1. *Any point in the line satisfies the given condition: and any point not in the line does not satisfy the given condition.*

Or, to prove

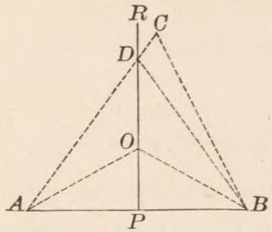
2. *Any point that satisfies the given condition lies in the line; and any point in the line satisfies the given condition.*

Note. The word *locus* (pronounced lo′kus) is a Latin word that signifies *place.* The plural of locus is loci (pronounced lo′si).

159. Def. A line which bisects a given line and is perpendicular to it is called the **perpendicular bisector** of the line.

PROPOSITION XXX. THEOREM.

160. *The perpendicular bisector of a given line is the locus of points equidistant from the extremities of the line.*



Let PR be the perpendicular bisector of the line AB, O any point in PR, and C any point not in PR.

Draw OA and OB, CA and CB.

To prove OA and OB equal, CA and CB unequal.

Proof. 1. $\triangle OPA = \triangle OPB$, § 144

for $PA = PB$ by hypothesis, and OP is common,

(*two right* $\triangle$s *are equal if their legs are equal, each to each*).

$\therefore OA = OB$. § 128

2. Since C is not in the $\perp$, CA or CB will cut the $\perp$.

Let CA cut the $\perp$ at D, and draw DB.

Then, by the first part of the proof $DA = DB$.

But $CB < CD + DB$. § 138

$\therefore CB < CD + DA$.

That is, $CB < CA$.

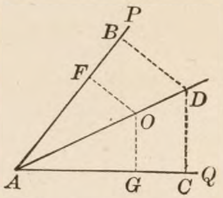
$\therefore PR$ is the locus of points equidistant from A and B. § 158, 1

Q.E.D.

161. COR. *Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line.*

PROPOSITION XXXI. THEOREM.

162. *The bisector of a given angle is the locus of points equidistant from the sides of the angle.*



Let O be any point equidistant from the sides of the angle PAQ.

To prove that O is in the bisector of the $\angle PAQ$.

Proof. Draw AO.

Suppose OF drawn $\perp$ to AP and $OG \perp$ to AQ.

In the rt. $\triangle$ AFO and AGO,

$AO = AO$, Iden.

$OF = OG$, Hyp.

$\therefore \triangle AFO = \triangle AGO$. § 151

$\therefore \angle FAO = \angle GAO$. § 128

$\therefore$ O is in the bisector of the $\angle PAQ$.

Let D be any point in the bisector of the angle PAQ.

To prove that D is equidistant from AP and AQ.

Proof. Suppose DB drawn $\perp$ to AP and $DC \perp$ to AQ.

In the rt. $\triangle$ ABD and ACD,

$AD = AD$, Iden.

$\angle DAB = \angle DAC$, Hyp.

$\therefore \triangle ABD = \triangle ACD$. § 141

$\therefore DB = DC$. § 128

$\therefore$ D is equidistant from AP and AQ.

$\therefore$ the bisector of the $\angle PAQ$ is the locus of points that are equidistant from its sides. § 158, 2

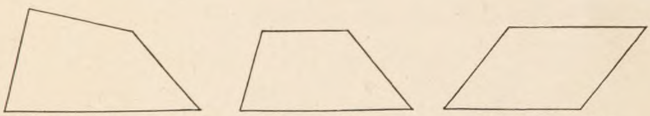
QUADRILATERALS.

163. A **quadrilateral** is a portion of a plane bounded by four straight lines. The bounding lines are the **sides**, the angles formed by these sides are the **angles**, and the vertices of these angles are the **vertices**, of the quadrilateral.

164. A **trapezium** is a quadrilateral which has no two sides parallel.

165. A **trapezoid** is a quadrilateral which has two sides, and only two sides, parallel.

166. A **parallelogram** is a quadrilateral which has its opposite sides parallel.



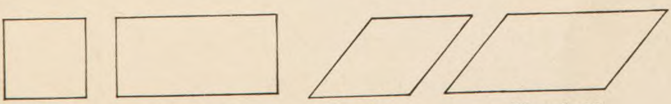
Trapezium. Trapezoid. Parallelogram.

167. A **rectangle** is a parallelogram which has its angles right angles.

168. A **square** is a rectangle which has its sides equal.

169. A **rhomboid** is a parallelogram which has its angles oblique angles.

170. A **rhombus** is a rhomboid which has its sides equal.

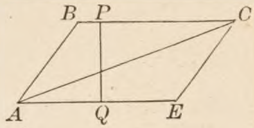


Square. Rectangle. Rhombus. Rhomboid.

171. The side upon which a parallelogram stands, and the opposite side, are called its lower and upper *bases*.

172. Two parallel sides of a trapezoid are called its **bases**, the other two sides its **legs**, and the line joining the middle points of the legs is called the **median** of the trapezoid.

173. A trapezoid is called an **isosceles trapezoid** if its legs are equal.

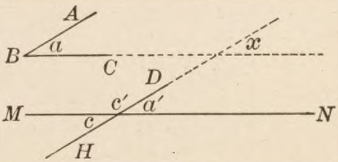


174. The **altitude** of a parallelogram or trapezoid is the perpendicular distance between its bases, as PQ.

175. A **diagonal** of a quadrilateral is a straight line joining two opposite vertices, as AC.

Proposition XXXII. Theorem.

176. *Two angles whose sides are parallel, each to each, are either equal or supplementary.*



Let BA be parallel to HD, and BC be parallel to MN.

To prove $\angle$s a, a' *and* c *equal;* a *and* c' *supplementary.*

Proof. Let HD and BC prolonged intersect at x.

Then $\angle a = \angle x$, and $\angle a' = \angle x$. § 112

$\therefore \angle a = \angle a'$. Ax. 1

Also $\angle c = \angle a'$ (§ 93). $\therefore \angle c = \angle a$. Ax. 1

Now $\angle a'$ and $\angle c'$ are supplementary. § 89

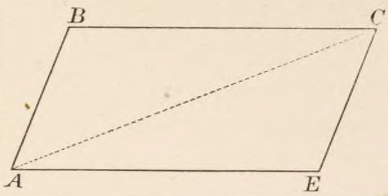
Put $\angle a$ for its equal, $\angle a'$.

Then $\angle a$ and $\angle c'$ are supplementary. Q.E.D.

177. Cor. *The opposite angles of a parallelogram are equal, and the adjacent angles are supplementary.*

PROPOSITION XXXIII. THEOREM.

178. *The opposite sides of a parallelogram are equal.*



Let the figure ABCE be a parallelogram.

To prove $BC = AE$, *and* $AB = EC$.

Proof. Draw the diagonal AC.

$\triangle ABC = \triangle CEA$. § 139

For AC is common,

$\angle BAC = \angle ACE$, and $\angle ACB = \angle CAE$, § 110

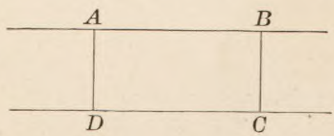
(being alt.-int. ▵ *of* ‖ *lines).*

$\therefore BC = AE$, and $AB = CE$, § 128

(being homologous sides of equal ▵*).* Q. E. D.

179. COR. 1. *A diagonal divides a parallelogram into two equal triangles.*

180. COR. 2. *Parallel lines comprehended between parallel lines are equal.*

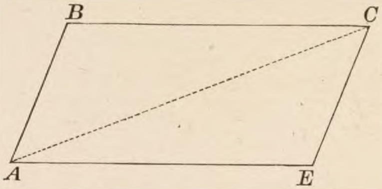


181. COR. 3. *Two parallel lines are everywhere equally distant.*

For if AB and DC are parallel, ⊥s dropped from *any* points in AB to DC, are equal, § 180. Hence, *all* points in AB are equidistant from DC.

Proposition XXXIV. Theorem.

182. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Let the figure ABCE be a quadrilateral, having BC equal to AE and AB to EC.

To prove that the figure $ABCE$ is a ▱.

Proof. Draw the diagonal AC.

In the $\triangle$s ABC and CEA,

$BC = AE$, Hyp.

$AB = CE$, Hyp.

and $AC = AC$. Iden.

$\therefore \triangle ABC = \triangle CEA$, § 150

(*having three sides of the one equal, respectively, to three sides of the other*).

$\therefore \angle ACB = \angle CAE$, § 128

and $\angle BAC = \angle ACE$,

(*being homologous* $\angle$s *of equal* $\triangle$s).

$\therefore BC$ is $\parallel$ to AE,

and AB is $\parallel$ to EC, § 111

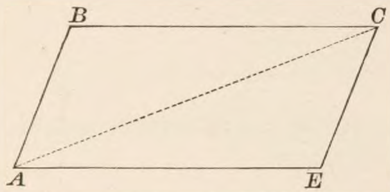
(*two lines in the same plane cut by a transversal are parallel, if the alt.-int.* $\angle$s *are equal*).

$\therefore$ the figure $ABCE$ is a ▱, § 166

(*having its opposite sides parallel*). Q. E. D.

PROPOSITION XXXV. THEOREM.

183. *If two sides of a quadrilateral are equal and parallel, then the other two sides are equal and parallel, and the figure is a parallelogram.*



Let the figure ABCE be a quadrilateral, having the side AE equal and parallel to BC.

To prove that AB is equal and parallel to EC.

Proof. Draw AC.

The $\triangle$ ABC and CEA are equal, § 143

(*having two sides and the included* $\angle$ *of each equal, respectively*).

For AC is common,

$BC = AE$, Hyp.

and $\angle BCA = \angle CAE$, § 110

(*being alt.-int.* $\angle$ *of* $\parallel$ *lines*).

$\therefore AB = EC$,

and $\angle BAC = \angle ACE$, § 128

(*being homologous parts of equal* $\triangle$).

$\therefore AB$ is $\parallel$ to EC, § 111

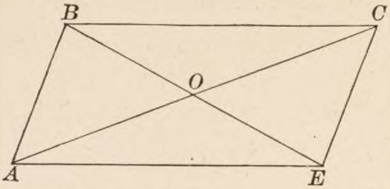
(*two lines are* $\parallel$, *if the alt.-int.* $\angle$ *are equal*).

$\therefore$ the figure $ABCE$ is a $\square$, § 166

(*the opposite sides being parallel*). Q. E. D.

Proposition XXXVI. Theorem.

184. *The diagonals of a parallelogram bisect each other.*



Let the figure ABCE be a parallelogram, and let the diagonals AC and BE cut each other at O.

To prove that $AO = OC$, *and* $BO = OE$.

Proof. In the $\triangle$ AOE and COB,

$AE = BC$, § 178

(*being opposite sides of a* $\square$).

$\angle OAE = \angle OCB$, § 110

and $\angle OEA = \angle OBC$,

(*being alt.-int.* $\angle$s *of* $\parallel$ *lines*).

$\therefore \triangle AOE = \triangle COB$, § 139

(*having two* $\angle$s *and the included side of the one equal, respectively, to two* $\angle$s *and the included side of the other*).

$\therefore AO = OC$, and $BO = OE$, § 128

(*being homologous sides of equal* $\triangle$s). Q. E. D.

Ex. 13. The median from the vertex to the base of an isosceles triangle is perpendicular to the base, and bisects the vertical angle.

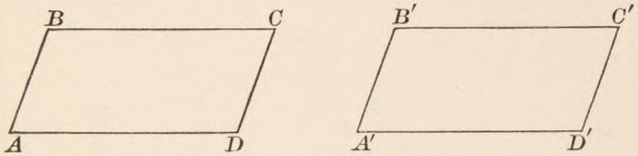
Ex. 14. If two straight lines are cut by a transversal so that the alternate-exterior angles are equal, the two straight lines are parallel.

Ex. 15. If two parallel lines are cut by a transversal, the two exterior angles on the same side of the transversal are supplementary.

Ex. 16. If two straight lines are cut by a transversal so as to make the exterior angles on the same side of the transversal supplementary, the two lines are parallel.

Proposition XXXVII. Theorem.

185. *Two parallelograms are equal, if two sides and the included angle of the one are equal, respectively, to two sides and the included angle of the other.*



In the parallelograms ABCD and A′B′C′D′, let AB be equal to A′B′, AD to A′D′, and angle A to A′.

To prove that the ▱s are equal.

Proof. Place the ▱ $ABCD$ on the ▱ $A'B'C'D'$, so that AD will fall on and coincide with its equal, $A'D'$.

Then AB will fall on $A'B'$, and B on B';

(*for* $\angle A = \angle A'$, *and* $AB = A'B'$, *by hyp.*).

Now, BC and $B'C'$ are both ∥ to $A'D'$ and drawn through B'.

∴ BC and $B'C'$ coincide, § 105

(*through a given point only one line can be drawn* ∥ *to a given line*).

Also DC and $D'C'$ are ∥ to $A'B'$ and drawn through D'.

∴ DC and $D'C'$ coincide. § 105

∴ C falls on C', § 48

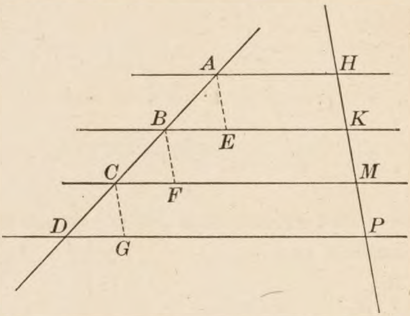
(*two lines can intersect in only one point*).

∴ the two ▱s coincide, and are equal. Q. E. D.

186. Cor. *Two rectangles having equal bases and altitudes are equal.*

Proposition XXXVIII. Theorem.

187. *If three or more parallels intercept equal parts on one transversal, they intercept equal parts on every transversal.*



Let the parallels AH, BK, CM, DP intercept equal parts HK, KM, MP on the transversal HP.

To prove that they intercept equal parts AB, BC, CD on the transversal AD.

Proof. Suppose AE, BF, and CG drawn ∥ to HP.

$\angle$s AEB, BFC, etc. $=$ $\angle$s HKE, KMF, etc., respectively. § 112

But $\angle$s HKE, KMF, etc. are equal. § 112

$\therefore$ $\angle$s AEB, BFC, etc. are equal. Ax. 1

Also $\angle$s BAE, CBF, etc. are equal. § 112

Now $AE = HK$, $BF = KM$, $CG = MP$, § 180

(*parallels comprehended between parallels are equal*).

$\therefore AE = BF = CG.$ Ax. 1

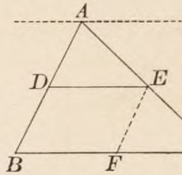
$\therefore \triangle ABE = \triangle BCF = \triangle CDG,$ § 139

(*having two* $\angle$s *and the included side of each respectively equal*).

$\therefore AB = BC = CD.$ § 128

Q.E.D.

188. Cor. 1. *If a line is parallel to the base of a triangle and bisects one side, it bisects the other side also.*

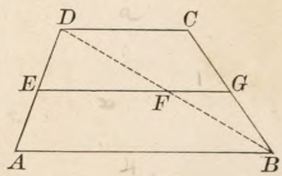


Let DE be $\parallel$ to BC and bisect AB. Suppose a line is drawn through A $\parallel$ to BC. Then this line is $\parallel$ to DE, by § 106. The three parallels by hypothesis intercept equal parts on the transversal AB, and therefore, by § 187, they intercept equal parts on the transversal AC; that is, the line DE bisects AC.

189. Cor. 2. *The line which joins the middle points of two sides of a triangle is parallel to the third side, and is equal to half the third side.*

A line drawn through D, the middle point of AB, $\parallel$ to BC, passes through E, the middle point of AC, by § 188. Therefore, the line joining D and E coincides with this parallel and is $\parallel$ to BC. Also, since EF drawn $\parallel$ to AB bisects AC, it bisects BC, by § 188; that is, $BF = FC = \frac{1}{2}BC$. But $BDEF$ is a $\square$ by § 166, and therefore $DE = BF = \frac{1}{2}BC$.

190. Cor. 3. *The median of a trapezoid is parallel to the bases, and is equal to half the sum of the bases.*



Draw the diagonal DB. In the $\triangle ADB$ join E, the middle point of AD, to F, the middle point of DB. Then, by § 189, EF is $\parallel$ to AB and $= \frac{1}{2}AB$. In the $\triangle DBC$ join F to G, the middle point of BC. Then FG is $\parallel$ to DC and $= \frac{1}{2}DC$. AB and FG, being $\parallel$ to DC, are $\parallel$ to each other. But only one line can be drawn through F $\parallel$ to AB (§ 105). Therefore FG is the prolongation of EF. Hence, EFG is parallel to AB and DC, and equal to $\frac{1}{2}(AB + DC)$.

POLYGONS IN GENERAL.

191. A **polygon** is a portion of a plane bounded by straight lines.

The bounding lines are the **sides**, and their sum, the **perimeter** of the polygon. The angles included by the adjacent sides are the **angles** of the polygon, and the vertices of these angles are the **vertices** of the polygon. The number of sides of a polygon is evidently equal to the number of its angles.

192. A **diagonal** of a polygon is a line joining the vertices of two angles not adjacent; as, AC (Fig. 1).

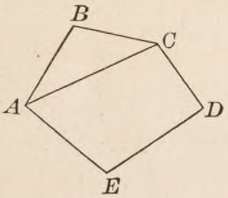


FIG. 1.

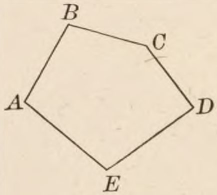


FIG. 2.

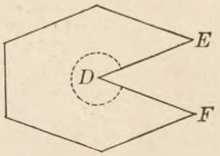


FIG. 3.

193. An **equilateral polygon** is a polygon which has all its sides equal.

194. An **equiangular polygon** is a polygon which has all its angles equal.

195. A **convex polygon** is a polygon of which no side, when produced, will enter the polygon.

196. A **concave polygon** is a polygon of which two or more sides, if produced, will enter the polygon.

197. Each angle of a convex polygon (Fig. 2) is called a *salient* angle, and is less than a straight angle.

198. The angle EDF of the concave polygon (Fig. 3) is called a *re-entrant* angle, and is greater than a straight angle.

When the term polygon is used, a *convex* polygon is meant.

199. Two polygons are *equal* when they can be divided by diagonals into the same number of triangles, equal each to each, and similarly placed; for if the polygons are applied to each other, the corresponding triangles will coincide, and hence the polygons will coincide and be equal.

200. Two polygons are *mutually equiangular*, if the angles of the one are equal to the angles of the other, each to each, when taken in the same order. Figs. 1 and 2.

201. The equal angles in mutually equiangular polygons are called *homologous* angles; and the sides which are included by homologous angles are called *homologous* sides.

202. Two polygons are *mutually equilateral*, if the sides of the one are equal to the sides of the other, each to each, when taken in the same order. Figs. 1 and 2.

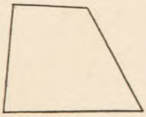


FIG. 4.

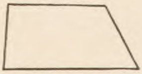


FIG. 5.


FIG. 6.

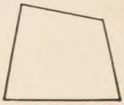


FIG. 7.

203. Two polygons may be mutually equiangular without being mutually equilateral; as, Figs. 4 and 5.

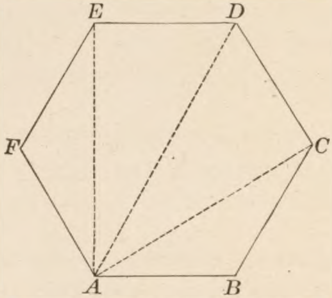
And, *except in the case of triangles*, two polygons may be mutually equilateral without being mutually equiangular; as, Figs. 6 and 7.

If two polygons are mutually equilateral and mutually equiangular, *they are equal*, for they can be made to coincide.

204. A polygon of three sides is called a *triangle;* one of four sides, a *quadrilateral;* one of five sides, a *pentagon;* one of six sides, a *hexagon;* one of seven sides, a *heptagon;* one of eight sides, an *octagon;* one of ten sides, a *decagon;* one of twelve sides, a *dodecagon.*

Proposition XXXIX. Theorem.

205. *The sum of the interior angles of a polygon is equal to two right angles, taken as many times less two as the figure has sides.*



Let the figure ABCDEF be a polygon, having n sides.

To prove that $\angle A + \angle B + \angle C$, etc. $= (n - 2)\, 2$ *rt.* $\angle s$.

Proof. From A draw the diagonals AC, AD, and AE.

The sum of the $\angle s$ of the $\triangle s$ is equal to the sum of the $\angle s$ of the polygon.

Now, there are $(n - 2)$ $\triangle s$,

and the sum of the $\angle s$ of each $\triangle = 2$ rt. $\angle s$. § 129

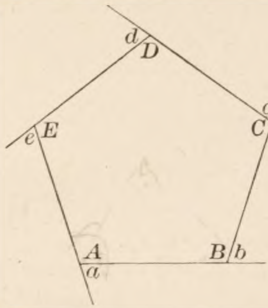
$\therefore$ the sum of the $\angle s$ of the $\triangle s$, that is, the sum of the $\angle s$ of the polygon is equal to $(n-2)\, 2$ rt. $\angle s$. Q.E.D.

206. Cor. *The sum of the angles of a quadrilateral equals 4 right angles; and if the angles are all equal, each is a right angle. In general, each angle of an equiangular polygon of n sides is equal to* $\frac{2(n-2)}{n}$ *right angles.*

Ex. 17. How many diagonals can be drawn in a polygon of n sides?

Proposition XL. Theorem.

207. *The exterior angles of a polygon, made by producing each of its sides in succession, are together equal to four right angles.*



Let the figure ABCDE be a polygon, having its sides produced in succession.

To prove the sum of the ext. $\angle s = 4$ *rt.* $\angle s$.

Proof. Denote the int. $\angle s$ of the polygon by A, B, C, D, E, and the corresponding ext. $\angle s$ by a, b, c, d, e.

$$\angle A + \angle a = 2 \text{ rt. } \angle s, \qquad \S\ 89$$

and $$\angle B + \angle b = 2 \text{ rt. } \angle s,$$

(*being sup.-adj.* $\angle s$).

In like manner each pair of adj. $\angle s = 2$ rt. $\angle s$.

$\therefore$ the sum of the interior and exterior $\angle s$ of a polygon of n sides is equal to $2n$ rt. $\angle s$.

But the sum of the interior $\angle s = (n-2)\,2$ rt. $\angle s$, § 205

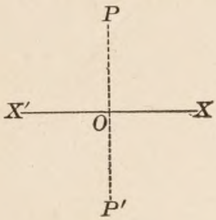
$$= 2n \text{ rt. } \angle s - 4 \text{ rt. } \angle s.$$

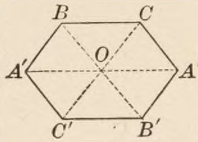
$\therefore$ the sum of the exterior $\angle s = 4$ rt. $\angle s$. Q.E.D.

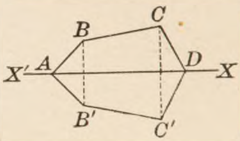
Ex. 18. How many sides has a polygon if the sum of its interior $\angle s$ is twice the sum of its exterior $\angle s$? ten times the sum of its exterior $\angle s$? n = 3

SYMMETRY.

208. Two points are said to be **symmetrical** with respect to a third point, called the **centre of symmetry**, if this third point bisects the straight line which joins them.







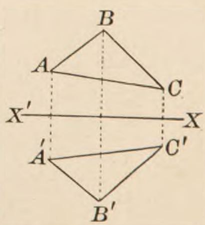
Two points are said to be *symmetrical* with respect to a straight line, called the **axis of symmetry**, if this straight line bisects at right angles the straight line which joins them.

Thus, P and P' are symmetrical with respect to O as a centre, and XX' as an axis, if O bisects the line PP', and if XX' bisects PP' at right angles.

209. A figure is symmetrical with respect to a point as a centre of symmetry, if the point bisects every straight line drawn through it and terminated by the boundary of the figure.

210. A figure is symmetrical with respect to a line as an axis of symmetry if one of the parts of the figure coincides, point for point, with the other part when it is folded over on that line as an axis.

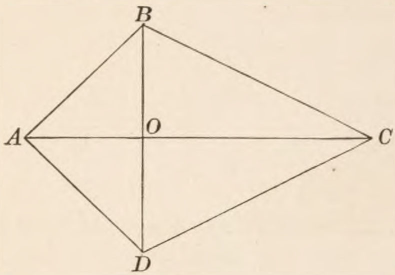
211. Two figures are said to be symmetrical with respect to an axis if every point of one has a corresponding symmetrical point in the other.



Thus, if every point in the figure $A'B'C'$ has a symmetrical point in ABC, with respect to XX' as an axis, the figure $A'B'C'$ is symmetrical to ABC with respect to XX' as an axis.

Proposition XLI. Theorem.

212. *A quadrilateral which has two adjacent sides equal, and the other two sides equal, is symmetrical with respect to the diagonal joining the vertices of the angles formed by the equal sides, and the diagonals are perpendicular to each other.*



Let ABCD be a quadrilateral, having AB equal to AD, and CB equal to CD, and having the diagonals AC and BD.

To prove that the diagonal AC is an axis of symmetry, and that it is $\perp$ to the diagonal BD.

Proof. In the $\triangle$ ABC and ADC,

$AB = AD$, and $BC = DC$, Hyp.

and $AC = AC$. Iden.

$\therefore \triangle ABC = \triangle ADC$. § 150

$\therefore \angle BAC = \angle DAC$, and $\angle BCA = \angle DCA$.

Hence, if ABC is turned on AC as an axis until it falls on ADC, AB will fall upon AD, CB on CD, and OB on OD.

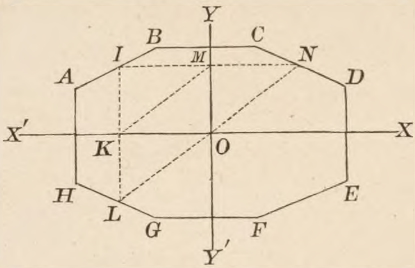
$\therefore$ the $\triangle ABC$ will coincide with the $\triangle ADC$.

$\therefore AC$ is an axis of symmetry (§ 210) and is $\perp$ to BD. § 208

Q. E. D.

PROPOSITION XLII. THEOREM.

213. *If a figure is symmetrical with respect to two axes perpendicular to each other, it is symmetrical with respect to their intersection as a centre.*



Let the figure ABCDEFGH be symmetrical with respect to the two perpendicular axes XX′, YY′, which intersect at O.

To prove that O is the centre of symmetry of the figure.

Proof. Let N be any point in the perimeter.

Suppose NMI drawn $\perp$ to YY', $IKL \perp$ to XX'.

Then NI is $\parallel$ to XX' and IL is $\parallel$ to YY'. § 104

Draw LO, ON, and KM.

Now $KI = KL$, § 208

(*the figure being symmetrical with respect to* XX').

But $KI = OM$. § 180

$\therefore KL = OM$, and $KLOM$ is a $\square$. § 183

$\therefore LO$ is equal and parallel to KM. § 183

In like manner ON is equal and parallel to KM.

$\therefore LON$ is a straight line. § 105

$\therefore O$ bisects LN, *any* straight line and therefore *every* straight line drawn through O and terminated by the perimeter.

$\therefore O$ is the centre of symmetry of the figure. Q. E. D.

REVIEW QUESTIONS ON BOOK I.

1. What is the subject-matter of Geometry ?

2. What is a geometric magnitude ?

3. What is an axiom ? a theorem ? a converse theorem ? an opposite theorem ? a contradictory theorem ?

4. Define a straight line ; a curved line ; a broken line ; a plane surface ; a curved surface.

5. How many points are necessary to determine a straight line ?

6. How many straight lines are necessary to determine a point ?

7. On what does the magnitude of an angle depend ?

8. Define a straight angle ; a right angle ; an oblique angle.

9. Define adjacent angles ; complementary angles ; supplementary angles ; conjugate angles.

10. Define parallel lines and give the axiom of parallels.

11. If two lines in the same plane are parallel and cut by a transversal, what pairs of angles are equal ? what pairs are supplementary ?

12. Define a right triangle ; an isosceles triangle ; a scalene triangle.

13. To how many right angles is the sum of the angles of a triangle equal ? the sum of the acute angles of a right triangle ?

14. To what angles is the exterior angle of a triangle equal ?

15. What is the test of equality of two geometric magnitudes ?

16. How does a reciprocal theorem differ from a converse theorem ?

17. State the three cases in which two triangles are equal.

18. State the cases in which two right triangles are equal.

19. What is meant by a locus of points ?

20. Where are the points located in a plane that are each equidistant from two given points ? from two intersecting lines ?

21. Define a parallelogram ; a trapezoid ; an isosceles trapezoid.

22. When is a figure symmetrical with respect to a centre ?

23. When is a figure symmetrical with respect to an axis ?

24. Must a triangle be equiangular if equilateral ? must a triangle be equilateral if equiangular ?

25. When are two polygons said to be mutually equiangular ?

26. When are two polygons said to be mutually equilateral ?

27. Can two polygons of more than three sides be mutually equiangular without being mutually equilateral ? mutually equilateral without being mutually equiangular ?

28. What line do two points each equidistant from the extremities of a given straight line determine ?

METHODS OF PROVING THEOREMS.

214. There are *three* general methods of proving theorems, the **synthetic,** the **analytic,** and the **indirect** methods.

The *synthetic* method is the method employed in most of the theorems already given, and consists in putting together known truths in order to obtain a new truth.

The *analytic* method is the reverse of the synthetic method. It asserts that the conclusion is true if another proposition is true, and so on step by step, until a known truth is reached. Thus, proposition A is true if proposition B is true, and B is true if C is true; but C *is* true, hence A and B are true.

If a known truth *suggests* the required proof, it is best to use the synthetic form at once. If no proof occurs to the mind, it is necessary to use the analytic method to *discover* the proof, and then the synthetic proof may be given.

The *indirect* method, or the method of *reductio ad absurdum,* is illustrated on page 41. It consists in proving a theorem to be true by proving its contradictory to be false.

215. Generally auxiliary lines are required, as a line *connecting two points;* a line *parallel to or perpendicular to a given line;* a line *produced by its own length;* a line *making with another line an angle equal to a given angle.*

Two lines are proved equal by proving them *homologous sides of equal triangles;* or *legs of an isosceles triangle;* or *opposite sides of a parallelogram.*

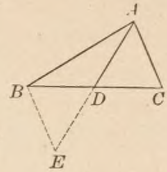
Two angles are proved equal by proving them *alternate-interior angles or exterior-interior angles of parallel lines;* or *homologous angles of equal triangles;* or *base angles of an isosceles triangle;* or *opposite angles of a parallelogram.*

Two suggestions are of special importance to the beginner:

1. *Draw as accurate figures as possible.*
2. *Draw as general figures as possible.*

Prove by the analytic method:

Ex. 19. *A median of a triangle is less than half the sum of the two adjacent sides.*



To prove the median $AD < \frac{1}{2}(AB + AC)$.

Now $AD < \frac{1}{2}(AB + AC)$,

if $2\,AD < AB + AC$.

This suggests producing AD by its own length to E, and joining BE.

Then $AE = 2\,AD$,

and $2\,AD < AB + AC$ if $AE < AB + AC$.

But $AE < AB + BE$. § 138

$\therefore AE < AB + AC$ if $AC = BE$.

And $AC = BE$ if $\triangle ACD = \triangle EBD$. § 128

But $\triangle ACD = \triangle EBD$. § 143

For $CD = DB$, Hyp.

$AD = DE$, Const.

and $\angle ADC = \angle BDE$. § 93

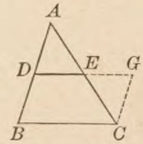
$\therefore AE < AB + AC$.

$\therefore AD < \frac{1}{2}(AB + AC)$.

Ex. 20. *A straight line which bisects two sides of a triangle is parallel to the third side.*

If $AD = DB$ and $AE = EC$, to prove $DE \parallel$ to BC.

Draw $CG \parallel$ to BA, and produce DE to meet it at G.



DE is $\parallel$ to BC if $BCGD$ is a $\square$. § 166

$BCGD$ is a $\square$ if $CG = BD$. § 183

$CG = BD$ if each is equal to AD. Ax. 1

Now $BD = AD$. Hyp.

And $CG = AD$ if $\triangle CGE = \triangle ADE$. § 128

But $\triangle CGE = \triangle ADE$. § 139

For $EC = AE$. Hyp.

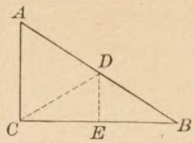
$\angle GEC = \angle AED$. § 93

$\angle ECG = \angle DAE$. § 110

$\therefore DE$ is $\parallel$ to BC.

Prove by the synthetic method:

Ex. 21. *The middle point of the hypotenuse of a right triangle is equidistant from the three vertices.*

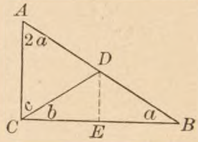


From D, the middle point, draw $DE \perp$ to CB.

DE is $\parallel$ to AC (why ?), and DE bisects CB (why ?).

$\therefore D$ is equidistant from B, A, and C. (Why ?)

Ex. 22. *If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter leg.*



The median $CD = BD = AD$ (Ex. 21).

Then $\angle b = \angle a$; and $\angle c = \angle 2a$. (Why ?)

Now $a + 2a = 90°$. (Why ?)

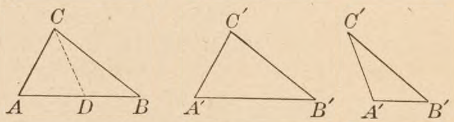
$\therefore \angle a = 30°$; $\angle 2a = 60°$; $\angle c = 60°$.

$\therefore \triangle ACD$ is equilateral (why ?), and AD, half of $AB = AC$. $\therefore AB = 2AC$.

Ex. 23. *If two triangles have two sides of the one equal, respectively, to two sides of the other, and the angles opposite two equal sides equal, the angles opposite the other two equal sides are equal or supplementary, and if equal the triangles are equal.*

Let $AC = A'C'$, $BC = B'C'$, and $\angle B = \angle B'$.

Place $\triangle A'B'C'$ on $\triangle ABC$ so that $B'C'$ shall coincide with BC, and $\angle A'$ and $\angle A$ shall be on the same side of BC.



Since $\angle B' = \angle B$, $B'A'$ will fall along BA, and A' will fall at A or at some other point in BA, as D. If A' falls at A, the $\triangle$s $A'B'C'$ and ABC coincide and are equal.

If A' falls at D, the $\triangle$s $A'B'C'$ and DBC coincide and are equal.

Since $CD = C'A' = CA$, $\angle A = \angle CDA$. (Why ?)

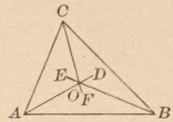
But $\angle$s CDA and CDB are supplements. (Why ?)

$\therefore \angle$s A and CDB are supplements. (Why ?)

Draw figures and show that the triangles are equal:

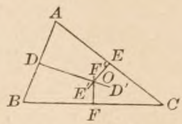
1. If the given angles B and B' are both right or both obtuse angles.
2. If the required angles A and A' are both acute, both right, or both obtuse.
3. If AC and $A'C'$ are not less than BC and $B'C'$, respectively.

Ex. 24. *The bisectors of the angles of a triangle meet in a point which is equidistant from the sides of the triangle.*



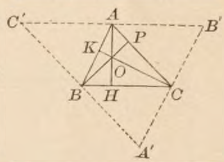
Let the bisectors AD and BE intersect at O. Then O being in AD is equidistant from AC and AB. (Why?) And O being in BE is equidistant from BC and AB. Hence, O is equidistant from AC and BC, and therefore in the bisector CF. (Why?)

Ex. 25. *The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the vertices of the triangle.*



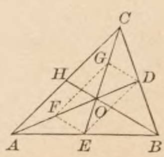
Let the $\perp$ bisectors EE' and DD' intersect at O. Then O being in EE' is equidistant from A and C. (Why?) And O being in DD' is equidistant from A and B. Hence, O is equidistant from B and C, and therefore is in the $\perp$ bisector FF'. (Why?)

Ex. 26. *The perpendiculars from the vertices of a triangle to the opposite sides meet in a point.*



Let the $\perp$s be AH, BP, and CK. Through A, B, C suppose $B'C'$, $A'C'$, $A'B'$, drawn $\parallel$ to BC, AC, AB, respectively. Then AH is $\perp$ to $B'C'$. (Why?) Now $ABCB'$ and $ACBC'$ are ▱s (why?), and $AB' = BC$, and $AC' = BC$. (Why?) That is, A is the middle point of $B'C'$. In the same way, B and C are the middle points of $A'C'$ and $A'B'$, respectively. Therefore, AH, BP, and CK are the $\perp$ bisectors of the sides of the $\triangle A'B'C'$. Hence, they meet in a point. (Why?)

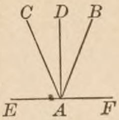
Ex. 27. *The medians of a triangle meet in a point which is two thirds of the distance from each vertex to the middle of the opposite side.*

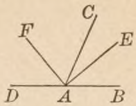


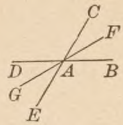
Let the two medians AD and CE meet in O. Take F the middle point of OA, and G of OC. Join GF, FE, ED, and DG. In $\triangle AOC$, GF is $\parallel$ to AC and equal to $\frac{1}{2}AC$. (Why?) DE is $\parallel$ to AC and equal to $\frac{1}{2}AC$. (Why?) Hence, $DGFE$ is a ▱. (Why?) Hence, $AF = FO = OD$, and $CG = GO = OE$. (Why?) Hence, *any median* cuts off *on any other median* two thirds of the distance from the vertex to the middle of the opposite side. Therefore, the median from B will cut off AO, two thirds of AD; that is, will pass through O.

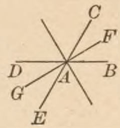
NOTE. If *three* or more lines pass through the same point, they are called *concurrent* lines.

Ex. 28. If an angle is bisected, and if a line is drawn through the vertex perpendicular to the bisector, this line forms equal angles with the sides of the given angle.









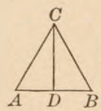
Ex. 29. The bisectors of two supplementary adjacent angles are perpendicular to each other.

Ex. 30. If the bisectors of two adjacent angles are perpendicular to each other, the adjacent angles are supplementary.

Ex. 31. The bisector of one of two vertical angles bisects the other.

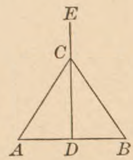
Ex. 32. The bisectors of two vertical angles form one line.

Ex. 33. The bisectors of the two pairs of vertical angles formed by two intersecting lines are perpendicular to each other.



Ex. 34. The bisector of the vertical angle of an isosceles triangle bisects the base, and is perpendicular to the base.

$\triangle ADC = \triangle BDC$ (§ 143).



Ex. 35. The perpendicular bisector of the base of an isosceles triangle passes through the vertex and bisects the angle at the vertex (§ 160).

Ex. 36. If the perpendicular bisector of the base of a triangle passes through the vertex, the triangle is isosceles.

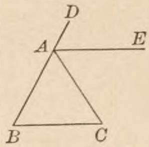
Ex. 37. Any point in the bisector of the vertical angle of an isosceles triangle is equidistant from the extremities of the base (Ex. 34, § 160).

Ex. 38. If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

Ex. 39. If two isosceles triangles are on the same base, a straight line passing through their vertices is perpendicular to the base, and bisects the base (§ 161).

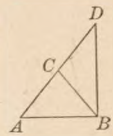
Ex. 40. Two isosceles triangles are equal when a side and an angle of the one are equal, respectively, to the homologous side and angle of the other.

Ex. 41. The bisector of an exterior angle of an isosceles triangle, formed by producing one of the legs through the vertex, is parallel to the base. Why does $\angle DAC = \angle B + \angle C$? Why does $\angle DAE = \angle ABC$? Why is $AE \parallel$ to BC?



Ex. 42. If the bisector of an exterior angle of a triangle is parallel to one side, the triangle is isosceles.

Ex. 43. If one of the legs of an isosceles triangle is produced through the vertex by its own length, the line joining the end of the leg produced to the nearer end of the base is perpendicular to the base.

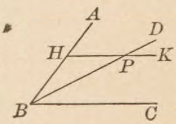


$$\angle CBA = \angle A, \text{ and } \angle CBD = \angle D. \quad \text{(Why ?)}$$
$$\therefore \angle ABD = \angle A + \angle D.$$

Ex. 44. A line drawn from the vertex of the right angle of a right triangle to the middle point of the hypotenuse divides the triangle into two isosceles triangles.

Ex. 45. If the equal sides of an isosceles triangle are produced through the vertex so that the external segments are equal, the extremities of these segments will be equally distant from the extremities of the base, respectively.

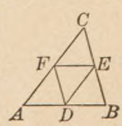
Ex. 46. If through any point in the bisector of an angle a line is drawn parallel to either of the sides of the angle, the triangle thus formed is isosceles.



Ex. 47. Through any point C in the line AB an intersecting line is drawn, and from any two points in this line equidistant from C perpendiculars are dropped on AB or AB produced. Prove that these perpendiculars are equal.

Ex. 48. If the median drawn from the vertex of a triangle to the base is equal to half the base, the vertical angle is a right angle.

Ex. 49. The lines joining the middle points of the sides of a triangle divide the triangle into four equal triangles.

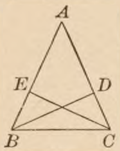


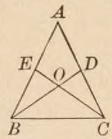
Ex. 50. The altitudes upon the legs of an isosceles triangle are equal.

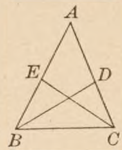
$$\text{Rt.}\ \triangle BEC = \text{rt.}\ \triangle CDB \ (\S\ 141).$$

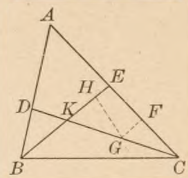
Ex. 51. If the altitudes upon two sides of a triangle are equal, the triangle is isosceles.

$$\text{Rt.}\ \triangle BEC \doteq \text{rt.}\ \triangle CDB \ (\S\ 151).$$









Ex. 52. The medians drawn to the legs of an isosceles triangle are equal.

$$\triangle BEC = \triangle CDB \ (\S\ 143).$$

Ex. 53. If the medians to two sides of a triangle are equal, the triangle is isosceles.

$$BO = CO, \text{ and } OE = OD \ (\text{Ex. } 27).$$

$$\angle BOE = \angle COD. \quad \therefore \triangle BOE = \triangle COD \ (\S\ 143).$$

Ex. 54. The bisectors of the base angles of an isosceles triangle are equal.

$$\triangle BEC = \triangle CDB \ (\S\ 139).$$

Ex. 55. Opposite Theorem. If a triangle is not isosceles, the bisectors of the base angles are not equal.

Let $\angle ABC$ be greater than $\angle ACB$; then $KC > KB$. (Why ?)

Now $CD > BE$, if KD is greater than or equal to KE.

But suppose $KD < KE$. Lay off $KH = KD$ and $KG = KB$, join HG, and draw $GF \parallel$ to BE.

$\triangle KDB = \triangle KHG$. (Why ?) $\therefore \angle KHG = \angle KDB$. (Why ?)

$\therefore \angle KEC$ is greater than $\angle KHG$. (Why ?) $\therefore GF > HE$. (Why ?)

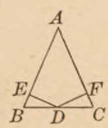
$\angle GFC$ is greater than $\angle FCG$ ($\frac{1}{2}\ ACB$). $\therefore CG > GF$, and $> HE$.

$\therefore KC - KG > KE - KH$, or $KC + KD > KB + KE$, or $CD > BE$.
or (KC + KG + ΔG) > KE − KH + BH)

Ex. 56. State the converse theorem of Ex. 54. Is the converse theorem true ?

Ex. 57. The perpendiculars dropped from the middle point of the base upon the legs of an isosceles triangle are equal.

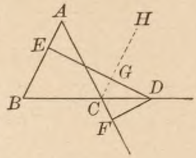
$$\triangle BED = \triangle CFD \ (\S\ 141).$$



Ex. 58. State and prove the converse.

$$\triangle BED = \triangle CFD \ (\S\ 151).$$

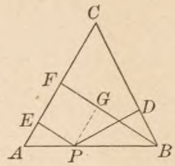
Ex. 59. The difference of the distances from any point in the base produced of an isosceles triangle to the equal sides of the triangle is constant.



Rt. $\triangle DGC =$ rt. $\triangle DFC$. (Why ?) $\therefore DF = DG$.

$\therefore DE - DF = DE - DG = EG$, the $\perp$ distance between the two ‖s, BA and CH.

Ex. 60. The sum of the perpendiculars dropped from any point in the base of an isosceles triangle to the legs is constant, and equal to the altitude upon one of the legs.



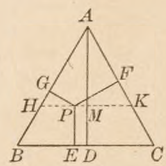
Let PE and PD be the $\perp$s and BF the altitude.

Draw $PG \perp$ to BF.

$EPGF$ is a parallelogram. (Why ?) $\therefore GF = PE$. It remains to prove $GB = PD$.

The rt. $\triangle PGB =$ the rt. $\triangle BDP$. (Why ?)

Ex. 61. The sum of the perpendiculars dropped from any point within an equilateral triangle to the three sides is constant, and equal to the altitude.

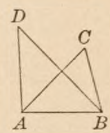


AD is the altitude, PE, PG, and PF the three perpendiculars. Through P draw HK ‖ to BC, meeting AD at M.

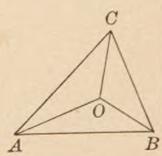
Then $\quad MD = PE$. (Why ?)

$$PG + PF = AM \text{ (Ex. 60).}$$

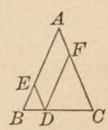
Ex. 62. ABC and ABD are two triangles on the same base AB, and on the same side of it, the vertex of each triangle being without the other. If AC equals AD, show that BC cannot equal BD (§ 154).



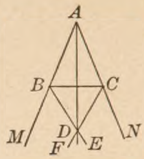
Ex. 63. The sum of the lines which join a point within a triangle to the three vertices is less than the perimeter, but greater than half the perimeter.



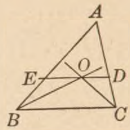
Ex. 64. If from any point in the base of an isosceles triangle parallels to the legs are drawn, a parallelogram is formed whose perimeter is constant, being equal to the sum of the legs of the triangle.



Ex. 65. The bisector of the vertical angle A of a triangle ABC, and the bisectors of the exterior angles at the base formed by producing the sides AB and AC, meet in a point which is equidistant from the base and the sides produced (§ 162).

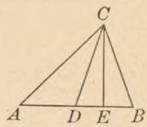


Ex. 66. If the bisectors of the base angles of a triangle are drawn, and through their point of intersection a line is drawn parallel to the base, the length of this parallel between the sides is equal to the sum of the segments of the sides between the parallel and the base.



$$\angle EOB = \angle OBC = \angle OBE. \quad \therefore BE = EO.$$

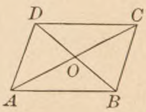
Ex. 67. The bisector of the vertical angle of a triangle makes with the perpendicular from the vertex to the base an angle equal to half the difference of the base angles.



Let $\angle B$ be greater than $\angle A$.

$$\angle DCE = 90° - \angle A - \angle ACD.$$
$$\angle ACD = 90° - \tfrac{1}{2}\angle A - \tfrac{1}{2}\angle B.$$
$$\therefore \angle DCE = 90° - \angle A - (90° - \tfrac{1}{2}\angle A - \tfrac{1}{2}\angle B) = \tfrac{1}{2}\angle B - \tfrac{1}{2}\angle A.$$

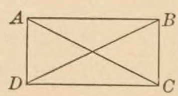
Ex. 68. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.



Prove $\triangle AOB = \triangle COD$.

Ex. 69. The diagonals of a rectangle are equal.

Prove $\triangle ABC = \triangle BAD$.

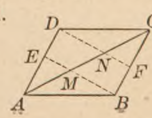


Ex. 70. If the diagonals of a parallelogram are equal, the figure is a rectangle.

Ex. 71. The diagonals of a rhombus are perpendicular to each other, and bisect the angles of the rhombus.

Ex. 72. The diagonals of a square are perpendicular to each other, and bisect the angles of the square.

Ex. 73. Lines from two opposite vertices of a parallelogram to the middle points of the opposite sides trisect the diagonal.



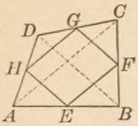
$EBFD$ is a ▱ (why ?), and DF is $\parallel$ to EB.

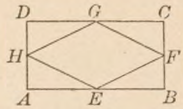
$AM = MN$, and $MN = CN$ (§ 188).

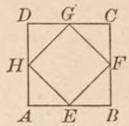
Ex. 74. The lines joining the middle points of the sides of any quadrilateral, taken in order, enclose a parallelogram.

Prove HG and $EF \parallel$ to AC; and FG and $EH \parallel$ to BD (§ 189).

Then HG and EF are each equal to $\frac{1}{2} AC$.







Ex. 75. The lines joining the middle points of the sides of a rhombus, taken in order, enclose a rectangle. (Proof similar to that of Ex. 74.)

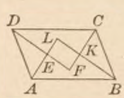
Ex. 76. The lines joining the middle points of the sides of a rectangle (not a square), taken in order, enclose a rhombus.

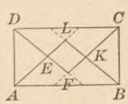
Ex. 77. The lines joining the middle points of the sides of a square, taken in order, enclose a square.

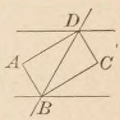
Ex. 78. The lines joining the middle points of the sides of an isosceles trapezoid, taken in order, enclose a rhombus or a square.

SHR and QFP drawn $\perp$ to AB are parallel. $\therefore PQSR$ is a $\square$, and by Const. is a rectangle or a square.

$\therefore EFGH$ is a rhombus or a square (Exs. 76, 77).







Ex. 79. The bisectors of the angles of a rhomboid enclose a rectangle.

Ex. 80. The bisectors of the angles of a rectangle enclose a square.

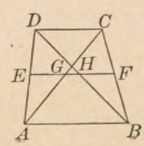
Ex. 81. If two parallel lines are cut by a transversal, the bisectors of the interior angles form a rectangle.

Ex. 82. The median of a trapezoid passes through the middle points of the two diagonals.

The median EF is $\parallel$ to AB and bisects AD (§ 190).

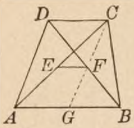
$\therefore$ it bisects DB.

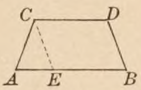
Likewise EF bisects BC and BD.

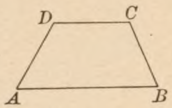


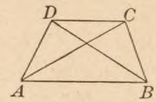
Ex. 83. The line joining the middle points of the diagonals of a trapezoid is equal to half the difference of the bases.

$$\triangle BFG = \triangle DFC. \quad \text{(Why ?)} \quad \therefore EF = \tfrac{1}{2} AG \ (\S\ 189).$$
$$CF = FG, \ DC = BG.$$
$$\therefore AG = AB - DC. \quad \therefore EF = \tfrac{1}{2}(AB - DC).$$









Ex. 84. In an isosceles trapezoid each base makes equal angles with the legs.

Draw $CE \parallel$ to DB. $CE = DB$. (Why ?) $\angle A = \angle CEA, \angle B = \angle CEA$. $\angle$s C and D have equal supplements.

Ex. 85. If the angles at the base of a trapezoid are equal, the other angles are equal, and the trapezoid is isosceles.

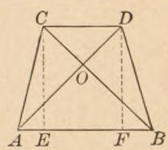
Ex. 86. In an isosceles trapezoid the opposite angles are supplementary.

$$\angle C = \angle D \text{ (Ex. 84).}$$

Ex. 87. The diagonals of an isosceles trapezoid are equal.

Prove $\triangle ACD = \triangle BDC$.

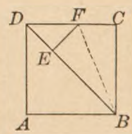
Ex. 88. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.



Draw CE and $DF \perp$ to AB.

$$\triangle ADF = \triangle BCE. \quad \text{(Why ?)}$$
$$\therefore \angle DAF = \angle CBA.$$
$$\triangle ABC = \triangle BAD.$$

Ex. 89. If from the diagonal DB, of a square $ABCD$, BE is cut off equal to BC, and EF is drawn perpendicular to BD meeting DC at F, then DE is equal to EF and also to FC.



$\angle EDF = 45°$, and $\angle DFE = 45°$; and $DE = EF$.
Rt. $\triangle BEF =$ rt. $\triangle BCF$ (§ 151); and $EF = FC$.

Ex. 90. Two angles whose sides are perpendicular, each to each, are either equal or supplementary.

Book II.

THE CIRCLE.

DEFINITIONS.

216. A **circle** is a portion of a plane bounded by a curved line, all points of which are equally distant from a point within called the **centre.** The bounding line is called the **circumference** of the circle.

217. A **radius** is a straight line from the centre to the circumference; and a **diameter** is a straight line through the centre, with its ends in the circumference.

By the definition of a circle, *all its radii are equal.* All its diameters are equal, since a diameter is equal to two radii.

218. Postulate. A circumference can be described from any point as a centre, with any given radius.

219. A **secant** is a straight line of unlimited length which intersects the circumference in two points; as, AD (Fig. 1).

220. A **tangent** is a straight line of unlimited length which has one point, and only one, in common with the circumference; as, BC (Fig. 1). In this case the circle is said to be tangent to the straight line. The common point is called the **point of contact,** or **point of tangency.**

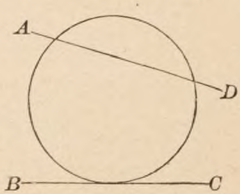


FIG. 1.

221. Two *circles* are tangent to each other, if both are tangent to a straight line at the same point; and are said to be tangent *internally* or *externally,* according as one circle lies wholly *within* or *without* the other.

222. An **arc** is any part of the circumference; as, BC (Fig. 3). Half a circumference is called a **semicircumference**. Two arcs are called **conjugate arcs**, if their sum is a circumference.

223. A **chord** is a straight line that has its extremities in the circumference; as, the straight line BC (Fig. 3).

224. A chord subtends two conjugate arcs. If the arcs are unequal, the less is called the **minor** arc, and the greater the **major** arc. A minor arc is generally called simply an arc.

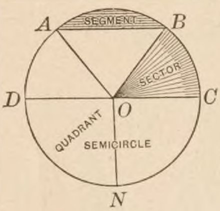


FIG. 2.

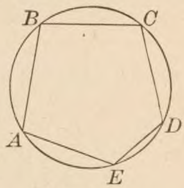


FIG. 3.

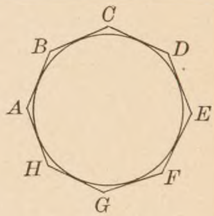


FIG. 4.

225. A **segment** of a circle is a portion of the circle bounded by an arc and its chord (Fig. 2).

226. A **semicircle** is a segment equal to half the circle (Fig. 2).

227. A **sector** of a circle is a portion of the circle bounded by two radii and the arc which they intercept. The angle included by the radii is called the *angle of the sector* (Fig. 2).

228. A **quadrant** is a sector equal to a quarter of the circle (Fig. 2).

229. An angle is called a **central angle**, if its vertex is at the centre and its sides are radii of the circle; as, $\angle AOD$ (Fig. 2).

230. An angle is called an **inscribed angle**, if its vertex is in the circumference and its sides are chords; as, $\angle ABC$ (Fig. 3).

An angle is *inscribed in a segment*, if its vertex is in the arc of the segment and its sides pass through the extremities of the arc.

231. A polygon is *inscribed in a circle,* if its sides are chords; and a circle is *circumscribed about a polygon,* if all the vertices of the polygon are in the circumference (Fig. 3).

232. A circle is *inscribed in a polygon,* if the sides of the polygon are tangent to the circle; and a polygon is *circumscribed about* a circle, if its sides are tangents (Fig. 4).

233. *Two circles are equal, if they have equal radii.*

For they will coincide, if their centres are made to coincide.

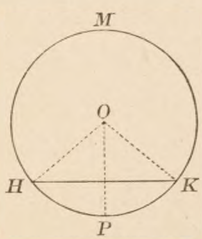
CONVERSELY: *Two equal circles have equal radii.*

234. *Two circles are concentric,* if they have the same centre.

ARCS, CHORDS, AND TANGENTS.

PROPOSITION I. THEOREM.

235. *A straight line cannot meet the circumference of a circle in more than two points.*



Let HK be any line meeting the circumference HKM in H and K.

To prove that HK cannot meet the circumference in any other point.

Proof. If possible, let HK meet the circumference in P.

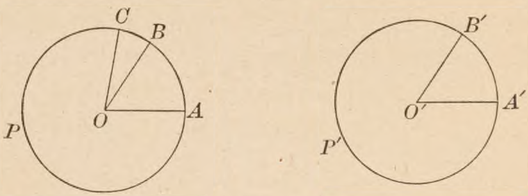
Then the radii OH, OP, and OK are equal. § 217

$\therefore P$ does not lie in the straight line HK. § 102

$\therefore HK$ meets the circumference in only two points. Q.E.D.

PROPOSITION II. THEOREM.

236. *In the same circle or in equal circles, equal central angles intercept equal arcs; and of two unequal central angles the greater intercepts the greater arc.*



In the equal circles whose centres are O and O′, let the angles AOB and A′O′B′ be equal, and angle AOC be greater than angle A′O′B′.

To prove that 1. *arc AB = arc $A'B'$;*
2. *arc $AC >$ arc $A'B'$.*

Proof. 1. Place the $\odot A'B'P'$ on the $\odot ABP$ so that the $\angle A'O'B'$ shall coincide with its equal, the $\angle AOB$.

Then A' falls on A, and B' on B. § 233

$\therefore$ arc $A'B'$ coincides with arc AB. § 216

2. Since the $\angle AOC$ is greater than the $\angle A'O'B'$, it is greater than the $\angle AOB$, the equal of the $\angle A'O'B'$.

Therefore, OC falls without the $\angle AOB$.

$\therefore$ arc $AC >$ arc AB. Ax. 8

$\therefore$ arc $AC >$ arc $A'B'$, the equal of arc AB. Q.E.D.

237. CONVERSELY: *In the same circle or in equal circles, equal arcs subtend equal central angles; and of two unequal arcs the greater subtends the greater central angle.*

To prove that 1. $\angle AOB = \angle A'O'B'$;

2. $\angle AOC$ *is greater than* $\angle A'O'B'$.

Proof. 1. Place the $\odot A'B'P'$ on the $\odot ABP$ so that $O'A'$ shall fall on its equal OA, and the arc $A'B'$ on its equal AB.

Then $O'B'$ will coincide with OB. § 47

$\therefore \angle A'O'B' = \angle AOB$. § 60

2. Since arc $AC > A'B'$, it is greater than arc AB, the equal of $A'B'$, and OB will fall within the $\angle AOC$.

$\therefore \angle AOC$ is greater than $\angle AOB$. Ax. 8

$\therefore \angle AOC$ is greater than $\angle A'O'B'$.

Q. E. D.

238. Cor 1. *In the same circle or in equal circles, two sectors that have equal angles are equal; two sectors that have unequal angles are unequal, and the greater sector has the greater angle.*

239. Cor. 2. *In the same circle or in equal circles, equal sectors have equal angles; and of two unequal sectors the greater has the greater angle.*

240. **Law of Converse Theorems.** It was stated in § 32 that the converse of a theorem is not necessarily true. If, however, a theorem is in fact a group of three theorems, and if *one of the hypotheses* of the group *must* be true, and *no two of the conclusions can be true at the same time*, then the converse of the theorem is *necessarily* true.

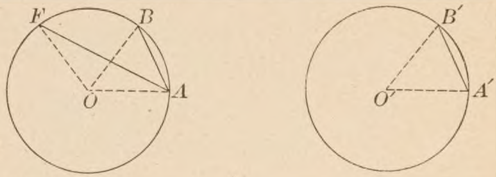
Proposition II. is a group of three theorems. It asserts that the arc AB is equal to the arc $A'B'$, if the angle AOB is equal to the angle $A'O'B'$; that the arc AB is greater than the arc $A'B'$, if the angle AOB is greater than the angle $A'O'B'$; that the arc AB is less than the arc $A'B'$, if the angle AOB is less than the angle $A'O'B'$.

One of these hypotheses must be true; for the angle AOB must be equal to, greater than, or less than, the angle $A'O'B'$.

No two of the conclusions can be true at the same time, for the arc AB cannot be both equal to and greater than the arc $A'B'$; nor can it be both equal to and less than the arc $A'B'$; nor both greater than and less than the arc $A'B'$. In such a case, the converse theorem is *necessarily* true, and no proof like that given in the text is required to establish it.

PROPOSITION III. THEOREM.

241. *In the same circle or in equal circles, equal arcs are subtended by equal chords; and of two unequal arcs the greater is subtended by the greater chord.*



In the equal circles whose centres are O and O′, let the arcs AB and A′B′ be equal, and the arc AF greater than arc A′B′.

To prove that 1. *chord* $AB =$ *chord* $A'B'$;
2. *chord* $AF >$ *chord* $A'B'$.

Proof. Draw the radii OA, OB, OF, $O'A'$, $O'B'$.

1. The $\triangle AOB$ and $A'O'B'$ are equal. § 143

For $OA = O'A'$, and $OB = O'B'$, § 233

(*radii of equal circles*),

and $\angle AOB = \angle A'O'B'$, § 237

(*in equal* ⓢ *equal arcs subtend equal central* ∠s).

$\therefore$ chord $AB =$ chord $A'B'$. § 128

2. In the $\triangle AOF$ and $A'O'B'$,

$OA = O'A'$, and $OF = O'B'$. § 233

But the $\angle AOF$ is greater than the $\angle A'O'B'$, § 237

(*in equal* ⓢ, *the greater of two unequal arcs subtends the greater* ∠).

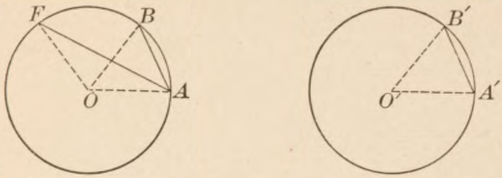
$\therefore$ chord $AF >$ chord $A'B'$. § 154

Q. E. D.

242. COR. *In the same circle or in equal circles, the greater of two unequal major arcs is subtended by the less chord.*

PROPOSITION IV. THEOREM.

243. CONVERSELY: *In the same circle or in equal circles, equal chords subtend equal arcs; and of two unequal chords the greater subtends the greater arc.*



In the equal circles whose centres are O and O′, let the chords AB and A′B′ be equal, and the chord AF greater than A′B′.

To prove that 1. *arc* $AB =$ *arc* $A'B'$;
2. *arc* $AF >$ *arc* $A'B'$.

Proof. Draw the radii OA, OB, OF, $O'A'$, $O'B'$.

1. The ⦻ OAB and $O'A'B'$ are equal. § 150

For $OA = O'A'$, and $OB = O'B'$, § 233

and chord $AB =$ chord $A'B'$. Hyp.

$\therefore \angle AOB = \angle A'O'B'$. § 128

$\therefore$ arc $AB =$ arc $A'B'$, § 236

(*in equal* ⓢ *equal central* ⦻ *intercept equal arcs*).

2. In the ⦻ OAF and $O'A'B'$,

$OA = O'A'$ and $OF = O'B'$. § 233

But chord $AF >$ chord $A'B'$. Hyp.

$\therefore$ the $\angle AOF$ is greater than the $\angle A'O'B'$. § 155

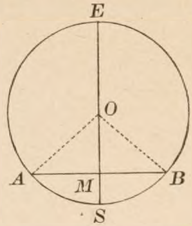
$\therefore$ arc $AF >$ arc $A'B'$, § 236

(*in equal* ⓢ *the greater central* $\angle$ *intercepts the greater arc*). Q.E.D.

244. COR. *In the same circle or in equal circles, the greater of two unequal chords subtends the less major arc.*

Proposition V. Theorem.

245. *A diameter perpendicular to a chord bisects the chord and the arcs subtended by it.*



Let ES be a diameter perpendicular to the chord AB at M.

To prove that $AM = BM$, $AS = BS$, *and* $AE = BE$.

Proof. Draw OA and OB from O, the centre of the circle.

	The rt. $\triangle$ OAM and OBM are equal.	§ 151
For	$OM = OM$,	Iden.
and	$OA = OB$.	§ 217
	$\therefore AM = BM$, and $\angle AOS = \angle BOS$.	§ 128
Likewise	$\angle AOE = \angle BOE$.	§ 85
	$\therefore AS = BS$, and $AE = BE$.	§ 236

Q. E. D

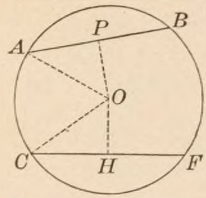
246. Cor. 1. *A diameter bisects the circumference and the circle.*

247. Cor. 2. *A diameter which bisects a chord is perpendicular to it.*

248. Cor. 3. *The perpendicular bisector of a chord passes through the centre of the circle, and bisects the arcs of the chord.*

Proposition VI. Theorem.

249. *In the same circle or in equal circles, equal chords are equally distant from the centre.* Conversely: *Chords equally distant from the centre are equal.*



Let AB and CF be equal chords of the circle ABFC.

To prove that AB and CF are equidistant from the centre O.

Proof. Draw $OP \perp$ to AB, $OH \perp$ to CF, and join OA and OC.

OP bisects AB, and OH bisects CF. § 245

The rt. $\triangle$ OPA and OHC are equal. § 151

For $AP = CH$, Ax. 7

and $OA = OC$. § 217

Hence, $OP = OH$. § 128

$\therefore AB$ and CF are equidistant from O.

Conversely: **Let $OP = OH$.**

To prove $AB = CF$.

Proof. The rt. $\triangle$ OPA and OHC are equal. § 151

For $OA = OC$, § 217

and $OP = OH$. Hyp.

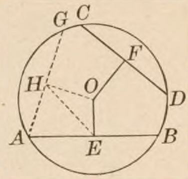
Hence, $AP = CH$. § 128

$\therefore AB = CF$. Ax. 6

Q.E.D.

PROPOSITION VII. THEOREM.

250. *In the same circle or in equal circles, if two chords are unequal, they are unequally distant from the centre; and the greater chord is at the less distance.*



In the circle whose centre is O, let the chords AB and CD be unequal, and AB the greater; and let OE be perpendicular to AB and OF perpendicular to CD.

To prove that $OE < OF$.

Proof. Suppose AG drawn equal to CD, and $OH \perp$ to AG.

Draw EH.

OE bisects AB, and OH bisects AG. § 245

By hypothesis, $AB > CD$.

$\therefore AB > AG$, the equal of CD.

$\therefore AE > AH$. Ax. 7

$\therefore \angle AHE$ is greater than $\angle AEH$. § 152

$\therefore \angle OHE$, the complement of $\angle AHE$, is less than $\angle OEH$, the complement of $\angle AEH$. Ax. 5

$\therefore OE < OH$. § 153

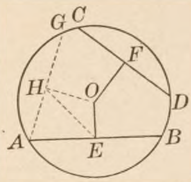
But $OH = OF$. § 249

$\therefore OE < OF$. Q. E. D.

Ex. 91. The perpendicular bisectors of the sides of an inscribed polygon are concurrent (pass through the same point).

PROPOSITION VIII. THEOREM.

251. CONVERSELY: *In the same circle or in equal circles, if two chords are unequally distant from the centre, they are unequal; and the chord at the less distance is the greater.*



In the circle whose centre is O, let AB and CD be unequally distant from O; and let OE, the perpendicular to AB, be less than OF, the perpendicular to CD.

To prove that $AB > CD$.

Proof. Suppose AG drawn equal to CD, and $OH \perp$ to AG.

Then $OH = OF$ § 249

Hence, $OE < OH$.

Draw EH.

$\angle OHE$ is less than $\angle OEH$. § 152

$\therefore \angle AHE$, the complement of $\angle OHE$, is greater than $\angle AEH$, the complement of $\angle OEH$. Ax. 5

$\therefore AE > AH$. § 153

But $AE = \frac{1}{2} AB$, and $AH = \frac{1}{2} AG$. § 245

$\therefore AB > AG$. Ax. 6

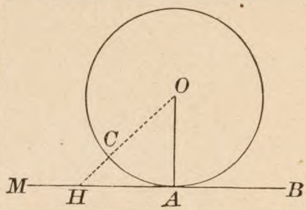
But $CD = AG$. Const.

$\therefore AB > CD$. Q.E.D.

252. COR. *A diameter of a circle is greater than any other chord.*

Proposition IX. Theorem.

253. *A straight line perpendicular to a radius at its extremity is a tangent to the circle.*



Let MB be perpendicular to the radius OA at A.

To prove that MB is a tangent to the circle.

Proof. From O draw *any other line* to MB, as OH.

Then $OH > OA$. § 97

$\therefore$ the point H is without the circle. § 216

Hence, *every point*, except A, of the line MB is without the circle, and therefore MB is a tangent to the circle at A. § 220

Q.E.D.

254. Cor. 1. *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

For OA is the shortest line from O to MB, and is therefore $\perp$ to MB (§ 98); that is, MB is $\perp$ to OA.

255. Cor. 2. *A perpendicular to a tangent at the point of contact passes through the centre of the circle.*

For a radius is $\perp$ to a tangent at the point of contact, and therefore a $\perp$ erected at the point of contact coincides with this radius and passes through the centre.

256. Cor. 3. *A perpendicular from the centre of a circle to a tangent passes through the point of contact.*

Proposition X. Theorem.

257. *Parallels intercept equal arcs on a circumference.*

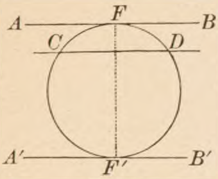


Fig. 1.

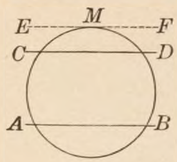


Fig. 2.

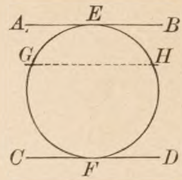


Fig. 3.

Case 1. **Let AB** (Fig. 1) **be a tangent at F parallel to CD, a secant.**

To prove that $\quad$ *arc* $CF =$ *arc* DF.

Proof. Suppose FF' drawn $\perp$ to AB.

Then FF' is a diameter of the circle. § 255

And FF' is also $\perp$ to CD. § 107

$\therefore CF = DF$, and $CF' = DF'$. § 245

Case 2. **Let AB and CD** (Fig. 2) **be parallel secants.**

To prove that $\quad$ *arc* $AC =$ *arc* BD.

Proof. Suppose $EF \parallel$ to CD and tangent to the circle at M.

Then arc AM = arc BM, Case 1

and arc CM = arc DM.

$\therefore$ arc AC = arc BD. Ax. 3

Case 3. **Let AB and CD** (Fig. 3) **be parallel tangents at E and F.**

To prove that $\quad$ *arc* $EGF =$ *arc* EHF.

Proof. Suppose GH drawn $\parallel$ to AB.

Then arc EG = arc EH, Case 1

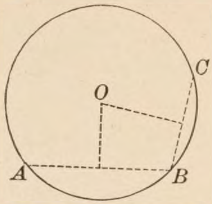
and arc GF = arc HF.

$\therefore$ arc EGF = arc EHF. Ax. 2

Q. E. D.

Proposition XI. Theorem.

258. *Through three points not in a straight line one circumference, and only one, can be drawn.*



Let A, B, C be three points not in a straight line.

To prove that one circumference, and only one, can be drawn through A, B, and C.

Proof. Draw AB and BC.

At the middle points of AB and BC suppose $\perp$s erected.

These $\perp$s will intersect at some point O, since AB and BC are not in the same straight line.

The point O is in the perpendicular bisector of AB, and is therefore equidistant from A and B; the point O is also in the perpendicular bisector of BC, and is therefore equidistant from B and C. § 160

Therefore, O is equidistant from A, B, and C; and a circumference described from O as a centre, with a radius OA, will pass through the three given points.

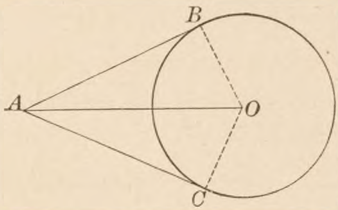
The centre of a circumference passing through the three points must be in both perpendiculars, and hence at their intersection. As two straight lines can intersect in only one point, O is the centre of the only circumference that can pass through the three given points. Q. E. D.

259. Cor. *Two circumferences can intersect in only two points.* For, if two circumferences have three points common, they coincide and form one circumference.

260. Def. **A tangent from an external point to a circle** is the part of the tangent between the external point and the point of contact.

Proposition XII. Theorem.

261. *The tangents to a circle drawn from an external point are equal, and make equal angles with the line joining the point to the centre.*



Let AB and AC be tangents from A to the circle whose centre is O, and let AO be the line joining A to the centre O.

To prove that $AB = AC$, *and* $\angle BAO = \angle CAO$.

Proof. Draw OB and OC.

AB is $\perp$ to OB, and $AC \perp$ to OC, § 254

(*a tangent to a circle is* $\perp$ *to the radius drawn to the point of contact*).

The rt. $\triangle$ OAB and OAC are equal. § 151

For OA is common, and the radii OB and OC are equal. § 217

$\therefore AB = AC$, and $\angle BAO = \angle CAO$. § 128

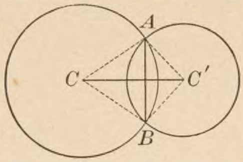
Q. E. D.

262. Def. The line joining the centres of two circles is called the **line of centres.**

263. Def. A tangent to two circles is called a **common external tangent** if it does not cut the line of centres, and a **common internal tangent** if it cuts the line of centres.

PROPOSITION XIII. THEOREM.

264. *If two circles intersect each other, the line of centres is perpendicular to their common chord at its middle point.*



Let C and C′ be the centres of the two circles, AB the common chord, and CC′ the line of centres.

To prove that CC' is $\perp$ to AB at its middle point.

Proof. Draw CA, CB, $C'A$, and $C'B$.

$CA = CB$, and $C'A = C'B$. § 217

$\therefore$ C and C' are two points, each equidistant from A and B.

$\therefore$ CC' is the perpendicular bisector of AB. § 161

Q. E. D.

Ex. 92. Describe the relative position of two circles if the line of centres :

(1) is greater than the sum of the radii;
(2) is equal to the sum of the radii;
(3) is less than the sum but greater than the difference of the radii;
(4) is equal to the difference of the radii;
(5) is less than the difference of the radii.

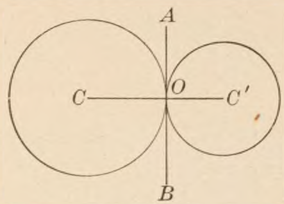
Illustrate each case by a figure.

Ex. 93. The straight line drawn from the middle point of a chord to the middle point of its subtended arc is perpendicular to the chord.

Ex. 94. The line which passes through the middle points of two parallel chords passes through the centre of the circle.

Proposition XIV. Theorem.

265. *If two circles are tangent to each other, the line of centres passes through the point of contact.*



Let the two circles, whose centres are C and C′, be tangent to the straight line AB at O, and CC′ the line of centres.

To prove that O is in the straight line CC'.

Proof. A $\perp$ to AB, drawn through the point O, passes through the centres C and C', § 255

(*a* $\perp$ *to a tangent at the point of contact passes through the centre of the circle*).

$\therefore$ the line CC', having two points in common with this $\perp$ must coincide with it. § 47

$\therefore$ O is in the straight line CC'. Q.E.D.

Ex. 95. Describe the relative position of two circles if they may have:

(1) two common external and two common internal tangents;

(2) two common external tangents and one common internal tangent;

(3) two common external tangents and no common internal tangent;

(4) one common external and no common internal tangent;

(5) no common tangent.

Illustrate each case by a figure.

Ex. 96. The line drawn from the centre of a circle to the point of intersection of two tangents is the perpendicular bisector of the chord joining the points of contact.

MEASUREMENT.

266. To **measure** a quantity of any kind is to find *the number of times* it contains a known quantity of the *same kind*, called the **unit of measure.**

The *number* which shows the number of times a quantity contains the unit of measure is called the **numerical measure** of that quantity.

267. No quantity is great or small except by comparison with another quantity of the *same kind*. This comparison is made by finding the numerical measures of the two quantities in terms of a common unit, and then dividing one of the measures by the other.

The quotient is called their **ratio**. In other words the ratio of two quantities of the same kind is the *ratio* of their *numerical measures* expressed in terms of a common unit.

The ratio of a to b is written $a : b$, or $\frac{a}{b}$.

268. Two quantities that can be expressed in *integers* in terms of a common unit are said to be **commensurable**, and the exact value of their ratio can be found. The common unit is called their *common measure*, and each quantity is called a *multiple* of this common measure.

Thus, a common measure of $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet is $\frac{1}{6}$ of a foot, which is contained 15 times in $2\frac{1}{2}$ feet, and 22 times in $3\frac{2}{3}$ feet. Hence, $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet are multiples of $\frac{1}{6}$ of a foot, since $2\frac{1}{2}$ feet may be obtained by taking $\frac{1}{6}$ of a foot 15 times, and $3\frac{2}{3}$ feet by taking $\frac{1}{6}$ of a foot 22 times. The ratio of $2\frac{1}{2}$ feet to $3\frac{2}{3}$ feet is expressed by the fraction $\frac{15}{22}$.

269. Two quantities of the same kind that cannot *both* be expressed in *integers* in terms of a common unit, are said to be **incommensurable**, and the *exact value* of their ratio cannot be found. But by taking the unit sufficiently small, an *approximate value* can be found that shall differ from the true value of the ratio by less than any assigned value, however small.

Thus, suppose the ratio, $\frac{a}{b} = \sqrt{2}$.

Now $\sqrt{2} = 1.41421356 \cdots$, a value greater than 1.414213, but less than 1.414214.

If, then, a *millionth part* of b is taken as the unit of measure, the value of $\frac{a}{b}$ lies between 1.414213 and 1.414214, and therefore differs from either of these values by less than 0.000001.

By carrying the decimal further, an approximate value may be found that will differ from the true value of the ratio by less than *a billionth, a trillionth, or any other assigned value.*

In general, if $\frac{a}{b} > \frac{m}{n}$ but $< \frac{m+1}{n}$, then the error in taking either of these values for $\frac{a}{b}$ is less than $\frac{1}{n}$, the difference between these two fractions. But by increasing n indefinitely, $\frac{1}{n}$ can be decreased indefinitely, and a value of the ratio can be found within any required degree of accuracy.

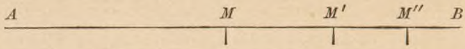
270. The ratio of two incommensurable quantities is called an **incommensurable ratio**; and is a *fixed value* which its successive approximate values constantly approach.

THE THEORY OF LIMITS.

271. When a quantity is regarded as having a *fixed* value throughout the same discussion, it is called a **constant**; but when it is regarded, under the conditions imposed upon it, as having *different successive* values, it is called a **variable.**

If a variable, by having different successive values, can be made to differ from a given constant by less than any assigned value, however small, but cannot be made absolutely equal to the constant, that constant is called the **limit** of the variable, and the variable is said to **approach the constant as its limit.**

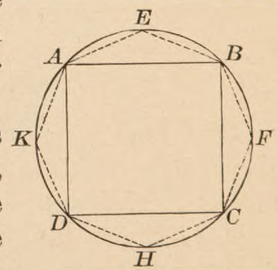
272. Suppose a point to move from A toward B, under the conditions that the first second it shall move one half the distance from A to B, that is, to M; the next second, one half the remaining distance, that is, to M'; and so on indefinitely.



Then it is evident that the moving point *may approach as near to B as we choose, but will never arrive at B.* For, however near it may be to B at any instant, the next second it will pass over half the distance still remaining; it must, therefore, approach nearer to B, since *half* the distance still remaining is *some* distance, but will not reach B, since *half* the distance still remaining is not the *whole* distance.

Hence, the distance from A to the moving point is an increasing variable, which indefinitely approaches the constant AB as its *limit;* and the distance from the moving point to B is a decreasing variable, which indefinitely approaches the *constant zero* as its *limit.*

273. Again, suppose a square $ABCD$ inscribed in a circle, and E, F, H, K the middle points of the arcs subtended by the sides of the square. If we draw the lines AE, EB, BF, etc., we shall have an inscribed polygon of double the number of sides of the square.



The length of the perimeter of this polygon, represented by the dotted lines, is greater than that of the square, since two sides replace each side of the square and form with it a triangle, and two sides of a triangle are together greater than the third side; but less than the length of the circumference, for it is made up of straight lines, each one of which is less than the part of the circumference between its extremities.

By continually doubling the number of sides of each resulting inscribed figure, the length of the perimeter will increase with the increase of the number of sides, but will not become equal to the length of the circumference.

The difference between the perimeter of the inscribed polygon and the circumference of the circle can be made less than any assigned value, but cannot be made equal to zero.

The length of the circumference is, therefore, the *limit* of the length of the perimeter as the *number of sides* of the inscribed figure is *indefinitely increased*. § 271

274. Consider the decimal 0.333 · · · which may be written

$$\tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000} + \cdots$$

The value of each fraction after the first is one tenth of the preceding fraction, and by continuing the series we shall reach a fraction less than *any* assigned value, that is, the values of the successive fractions *approach zero as a limit.*

The *sum* of these fractions is less than $\frac{1}{3}$; but the more terms we take, the nearer does the sum *approach* $\frac{1}{3}$ *as a limit.*

275. Test for a limit. In order to prove that a variable approaches a constant as a limit, it is necessary to prove that the difference between the variable and the constant:

1. *Can be made less than any assigned value, however small.*
2. *Cannot be made absolutely equal to zero.*

276. Theorem. *If the limit of a variable x is zero, then the limit of kx, the product of the variable by any finite constant k, is zero.*

1. Let q be any assigned quantity, however small.

Then $\frac{q}{k}$ is not 0. Hence x, which may differ as little as we please from 0, may be taken less than $\frac{q}{k}$, and then kx will be less than q.

2. Since x cannot be 0, kx cannot be 0.

Therefore, the limit of $kx = 0$. § 275

277. Cor. *If the limit of a variable x is zero, then the limit of the quotient of the variable by any finite constant k, is also zero.*

For $\frac{x}{k} = \frac{1}{k} \times x$, which by § 276 can be made less than any assigned value, however small, but cannot be made equal to zero.

278. Theorem. *The limit of the sum of a finite number of variables $x, y, z, \cdots$ is equal to the sum of their respective limits $a, b, c, \cdots$.*

Let $d, d', d'', \cdots$ denote the differences between $x, y, z, \cdots$ and $a, b, c, \cdots$, respectively. Then $d + d' + d'' + \cdots$ can be made less than any assigned quantity q.

For, if $d, d', d'', \cdots$ are n in number and d is the largest,

$$d + d' + d'' + \cdots < nd. \qquad (1)$$

Since d may be diminished at pleasure, we may make d so small that

$$d < \frac{q}{n}; \text{ and therefore } nd < q.$$

But by (1), $d + d' + d'' + \cdots < nd$, and therefore $< q$.

Therefore, the difference between $(x + y + z + \cdots)$ and $(a + b + c + \cdots)$ can be made less than any assigned quantity, but not zero.

Therefore, the limit of $(x + y + z + \cdots) = a + b + c + \cdots$. § 275

279. Theorem. *If the limit of a variable x is not zero, and if k is any finite constant, the limit of the product kx is equal to the limit of x multiplied by k.*

1. If a denotes the limit of x, then x cannot be equal to a. § 271

   Therefore, kx cannot be equal to ka.

2. The limit of $(a - x) = 0$. Hence, the limit of $ka - kx = 0$. § 276

   Therefore, the limit of $kx = ka$. § 275

280. Cor. *The quotient of the limit of a variable x by any finite constant k is the limit of x divided by k.*

For $\frac{x}{k} = \frac{1}{k} \times x$, and $\frac{\text{the limit of } x}{k} = \frac{1}{k} \times$ the limit of x.

281. Theorem. *The limit of the product of two or more variables is the product of their respective limits, provided no one of these limits is zero.*

If x and y are variables, a and b their respective limits, we may put $x = a - d$, $y = b - d'$; then d and d' are variables which can be made less than any assigned quantity, but not zero. § 275

Now,
$$xy = (a - d)(b - d')$$
$$= ab - ad' - bd + dd'.$$
$$\therefore ab - xy = ad' + bd - dd'.$$

Since every term on the right contains d or d', the whole right member can be made less than any assigned quantity, but not zero. § 278

Hence, $ab - xy$ can be made less than any assigned quantity, but not zero.

Therefore, the limit of $xy = ab$. § 275

Similarly, for three or more variables.

282. Cor. 1. *The limit of the nth power of a variable is the nth power of its limit.*

For the limit of the product of the variables $x, y, z, \cdots$ to n factors is the product of their respective limits, the constants $a, b, c, \cdots$ to n factors (§ 281). If the n factors $xyz \cdots$ are each equal to x, and the n factors $abc \cdots$ are each equal to a, we have $xyz \cdots = x^n$, and $abc \cdots = a^n$.

Therefore, the limit of $x^n = a^n$.

283. Cor. 2. *The limit of the nth root of a variable is the nth root of its limit.*

For if the limit of $x = a$, we may put this in the following form,

the limit of $\sqrt[n]{x^n} = \sqrt[n]{a^n}$;

that is, the limit of $\sqrt[n]{xxx \cdots \text{ to } n \text{ factors}}$ is $\sqrt[n]{aaa \cdots \text{ to } n \text{ factors}}$.

Now, $xxx \cdots$ is a variable since each factor is a variable,

and $aaa \cdots$ is a constant since each factor is a constant.

If we denote $xxx \cdots$ to n factors by the variable y, and $aaa \cdots$ to n factors by the constant b, we have

the limit of $\sqrt[n]{y} = \sqrt[n]{b}$.

284. Theorem. *If two variables are constantly equal, and each approaches a limit, the limits are equal.*

Let x and y be two variables, a and b their respective limits, d and d' the respective differences between the variables and their limits. Then, if the variables are *increasing* toward their limits

$$a = x + d, \text{ and } b = y + d'.$$

Since the equation $x = y$ is always true, we have by subtraction

$$a - b = d - d'.$$

Now, $d - d'$ can be made less than any assigned value, for d and d' can each be made less than any assigned value.

Since a and b are constants, $a - b$ is a constant; therefore, $d - d'$, which is equal to $a - b$, is a constant.

But the only constant which is less than any assigned value is 0. Therefore, $d - d' = 0$. Therefore, $a - b = 0$, and $a = b$.

If the variables x and y are *decreasing* toward their limits a and b, respectively, then

$$a = x - d \text{ and } b = y - d'.$$

Therefore, by subtraction

$$a - b = d' - d.$$

Therefore, by the same proof as for increasing variables

$$a = b.$$

285. Theorem. *If two variables have a constant ratio, and each approaches a limit that is not zero, the limits have the same ratio.*

Let x and y be two variables, a and b their respective limits.

Let $\qquad \frac{x}{y} = r$, a constant; then $x = ry$.

Since x and ry are two variables that are always equal,

the limit of x = the limit of ry. § 284

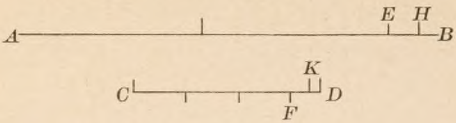
Now, the limit of $ry = r \times$ limit of y. § 279

But the limit of x is a, and the limit of y is b.

Therefore, $\qquad a = rb$; that is, $\frac{a}{b} = r$.

PROPOSITION XV. PROBLEM.

286. *To find the ratio of two straight lines.*



Let AB and CD be two straight lines.

To find the ratio of AB and CD.

Apply CD to AB as many times as possible.

Suppose twice, with a remainder EB.

Then apply EB to CD as many times as possible.

Suppose three times, with a remainder FD.

Then apply FD to EB as many times as possible.

Suppose once, with a remainder HB.

Then apply HB to FD as many times as possible.

Suppose once, with a remainder KD.

Then apply KD to HB as many times as possible.

Suppose KD is contained just twice in HB.

Then $HB = 2\,KD$;

$$FD = HB + KD = 3\,KD;$$
$$EB = FD + HB = 5\,KD;$$
$$CD = 3\,EB + FD = 18\,KD;$$
$$AB = 2\,CD + EB = 41\,KD.$$
$$\therefore \frac{AB}{CD} = \frac{41\,KD}{18\,KD} = \frac{41}{18}.$$

Q.E.F.

NOTE. By the same process the ratio of two arcs of the same circle or of equal circles can be found.

If the lines or arcs are incommensurable, an approximate value of the ratio can be found by the same method.

MEASURE OF ANGLES.

Proposition XVI. Theorem.

287. *In the same circle or in equal circles, two central angles have the same ratio as their intercepted arcs.*

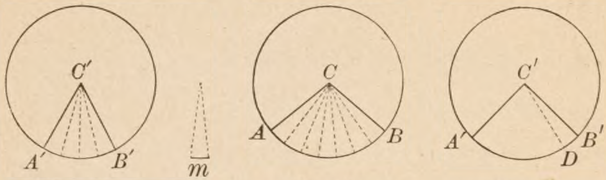


Fig. 1. Fig. 2. Fig. 3.

In the equal circles whose centres are C and C′, let ACB and A′C′B′ be the angles, AB and A′B′ the intercepted arcs.

To prove that $$\frac{\angle A'C'B'}{\angle ACB} = \frac{\text{arc } A'B'}{\text{arc } AB}.$$

Case 1. *When the arcs are commensurable* (Figs. 1 and 2).

Proof. Let the arc m be a common measure of $A'B'$ and AB.

Suppose m to be contained 4 times in $A'B'$,

and 7 times in AB.

Then $$\frac{\text{arc } A'B'}{\text{arc } AB} = \frac{4}{7}.$$

At the several points of division on AB and $A'B'$ draw radii.

These radii will divide $\angle ACB$ into 7 parts, and $\angle A'C'B'$ into 4 parts, equal each to each, § 237

(*in the same* ⊙, *or equal* Ⓢ, *equal arcs subtend equal central* $\angle$s).

$$\therefore \frac{\angle A'C'B'}{\angle ACB} = \frac{4}{7}.$$

$$\therefore \frac{\angle A'C'B'}{\angle ACB} = \frac{\text{arc } A'B'}{\text{arc } AB}. \qquad \text{Ax. 1}$$

CASE 2. *When the arcs are incommensurable* (Figs. 2 and 3).

Proof. Divide AB into any number of equal parts, and apply one of these parts to $A'B'$ as many times as $A'B'$ will contain it.

Since AB and $A'B'$ are incommensurable, a certain number of these parts will extend from A' to some point, as D, leaving a remainder DB' less than one of these parts. Draw $C'D$.

By construction AB and $A'D$ are commensurable.

$$\therefore \frac{\angle A'C'D}{\angle ACB} = \frac{\text{arc } A'D}{\text{arc } AB}. \qquad \text{Case 1}$$

By increasing the *number* of equal parts into which AB is divided we can diminish at pleasure the *length* of each part, and therefore make DB' less than any assigned value, however small, since DB' is always less than one of the equal parts into which AB is divided.

We cannot make DB' equal to zero, since, by hypothesis, AB and $A'B'$ are incommensurable. § 269

Hence, DB' approaches zero as a limit, if the number of parts of AB is indefinitely increased. § 275

And the corresponding angle $DC'B'$ approaches zero as a limit.

Therefore, the arc $A'D$ approaches the arc $A'B'$ as a limit, § 271 and the $\angle A'C'D$ approaches the $\angle A'C'B'$ as a limit.

Therefore, $\dfrac{\text{arc } A'D}{\text{arc } AB}$ approaches $\dfrac{\text{arc } A'B'}{\text{arc } AB}$ as a limit, § 280

and $\dfrac{\angle A'C'D}{\angle ACB}$ approaches $\dfrac{\angle A'C'B'}{\angle ACB}$ as a limit. § 280

But $\dfrac{\angle A'C'D}{\angle ACB}$ is constantly equal to $\dfrac{\text{arc } A'D}{\text{arc } AB}$, as $A'D$ varies in value and approaches $A'B'$ as a limit.

$$\therefore \frac{\angle A'C'B'}{\angle ACB} = \frac{\text{arc } A'B'}{\text{arc } AB}. \qquad \S\, 284$$

Q. E. D.

288. A circumference is divided into 360 equal parts, called *degrees;* and therefore a unit angle at the centre intercepts a unit arc on the circumference. Hence, the *numerical measure of a central angle* expressed in terms of the unit angle is equal to the *numerical measure of its intercepted arc* expressed in terms of the unit arc. This must be understood to be the meaning when it is said that

A central angle is measured by its intercepted arc.

Proposition XVII. Theorem.

289. *An inscribed angle is measured by half the arc intercepted between its sides.*

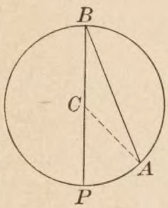


Fig. 1.

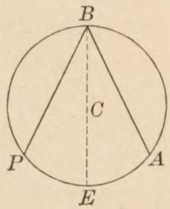


Fig. 2.

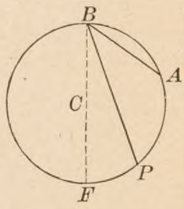


Fig. 3.

1. **Let the centre C (Fig. 1) be in one of the sides of the angle.**

To prove that the $\angle B$ *is measured by* $\frac{1}{2}$ *the arc* PA.

Proof. Draw CA.

$CA = CB$. § 217

$\therefore \angle B = \angle A$. § 145

But $\angle PCA = \angle B + \angle A$. § 137

$\therefore \angle PCA = 2 \angle B$.

But $\angle PCA$ is measured by arc PA, § 288

(*a central $\angle$ is measured by its intercepted arc*).

$\therefore \angle B$ is measured by $\frac{1}{2}$ arc PA.

2. **Let the centre C** (Fig. 2) **fall within the angle PBA.**

To prove that the $\angle PBA$ *is measured by* $\frac{1}{2}$ *the arc* PA.

Proof. Draw the diameter BCE.

Then $\angle EBA$ is measured by $\frac{1}{2}$ arc AE,

and $\angle EBP$ is measured by $\frac{1}{2}$ arc EP. Case 1

$\therefore \angle EBA + \angle EBP$ is measured by $\frac{1}{2}$ (arc AE + arc EP),

or $\angle PBA$ is measured by $\frac{1}{2}$ arc PA.

3. **Let the centre C** (Fig. 3) **fall without the angle PBA.**

To prove that the $\angle PBA$ *is measured by* $\frac{1}{2}$ *the arc* PA.

Proof. Draw the diameter BCF.

Then $\angle FBA$ is measured by $\frac{1}{2}$ arc FA,

and $\angle FBP$ is measured by $\frac{1}{2}$ arc FP. Case 1

$\therefore \angle FBA - \angle FBP$ is measured by $\frac{1}{2}$ (arc FA − arc FP),

or $\angle PBA$ is measured by $\frac{1}{2}$ arc PA. Q. E. D

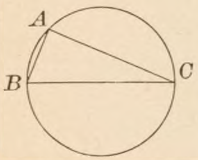


FIG. 4.

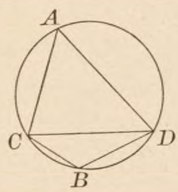


FIG. 5.

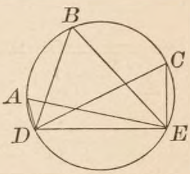


FIG. 6.

290. Cor. 1. *An angle inscribed in a semicircle is a right angle.* For it is measured by half a semicircumference (Fig. 4).

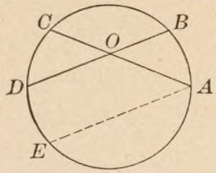
291. Cor. 2. *An angle inscribed in a segment greater than a semicircle is an acute angle.* For it is measured by an arc less than half a semicircumference; as, $\angle CAD$ (Fig. 5).

292. Cor. 3. *An angle inscribed in a segment less than a semicircle is an obtuse angle.* For it is measured by an arc greater than half a semicircumference; as, $\angle CBD$ (Fig. 5).

293. Cor. 4. *Angles inscribed in the same segment or in equal segments are equal* (Fig. 6).

Proposition XVIII. Theorem.

294. *An angle formed by two chords intersecting within the circumference is measured by half the sum of the intercepted arcs.*



Let the angle COD be formed by the chords AC and BD.

To prove that the $\angle\ COD$ *is measured by* $\frac{1}{2}(CD + AB)$.

Proof. Suppose AE drawn ∥ to BD.

Then arc $AB =$ arc DE, § 257

(*parallels intercept equal arcs on a circumference*).

Also $\angle\ COD = \angle\ CAE$, § 112

(*ext.-int.* ∠s *of* ∥s).

But $\angle\ CAE$ is measured by $\frac{1}{2}(CD + DE)$, § 289

(*an inscribed* ∠ *is measured by half its intercepted arc*).

Put $\angle\ COD$ for its equal, the $\angle\ CAE$, and arc AB for its equal, the arc DE; then $\angle\ COD$ is measured by $\frac{1}{2}(CD + AB)$.

Q. E. D.

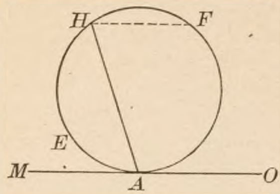
Ex. 97. The opposite angles of an inscribed quadrilateral are supplementary.

Ex. 98. If through a point within a circle two perpendicular chords are drawn, the sum of either pair of the opposite arcs which they intercept is equal to a semicircumference.

Ex. 99. The line joining the centre of the square described upon the hypotenuse of a right triangle to the vertex of the right angle bisects the right angle.

Proposition XIX. Theorem.

295. *An angle included by a tangent and a chord drawn from the point of contact is measured by half the intercepted arc.*



Let MAH be the angle included by the tangent MO to the circle at A and the chord AH.

To prove that the $\angle MAH$ *is measured by* $\frac{1}{2}$ *the arc* AEH.

Proof. Suppose HF drawn ∥ to MO.

Then arc $AF =$ arc AEH, § 257

(*parallels intercept equal arcs on a circumference*).

Also $\angle MAH = \angle AHF$, § 110

(*alt.-int.* ∠s *of* ∥s).

But $\angle AHF$ is measured by $\frac{1}{2}$ AF, § 289

(*an inscribed* ∠ *is measured by half its intercepted arc*).

Put $\angle MAH$ for its equal, the $\angle AHF$, and arc AEH for its equal, the arc AF; then $\angle MAH$ is measured by $\frac{1}{2}$ arc AEH.

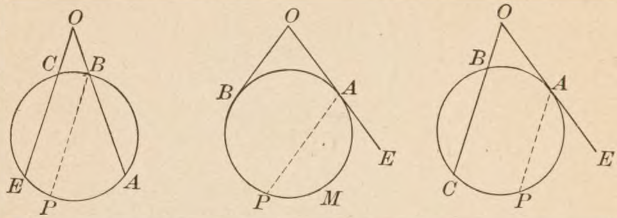
Likewise, the $\angle OAH$, the supplement of the $\angle MAH$, is measured by half the arc AFH, the conjugate of the arc AEH.

Q.E.D.

Ex. 100. Two circles are tangent externally at A, and a common external tangent touches them at B and C, respectively. Show that angle BAC is a right angle.

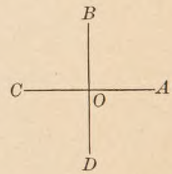
Proposition XX. Theorem.

296. *An angle formed by two secants, two tangents, or a tangent and a secant, drawn to a circle from an external point, is measured by half the difference of the intercepted arcs.*



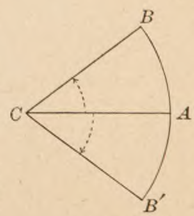
The proof of this theorem is left as an exercise for the student.

297. Positive and Negative Quantities. In measurements it is convenient to mark the distinction between two quantities that are measured in *opposite directions*, by calling one of them **positive** and the other **negative**.



Thus, if OA is considered positive, then OC may be considered negative, and if OB is considered positive, then OD may be considered negative.

When this distinction is applied to angles, an angle is considered to be *positive*, if the rotating line that describes it moves in the opposite direction to the hands of a clock (counter clockwise), and to be *negative*, if the rotating line moves in the same direction as the hands of a clock (clockwise).



Arcs corresponding to positive angles are considered *positive*, and arcs corresponding to negative angles are considered *negative*.

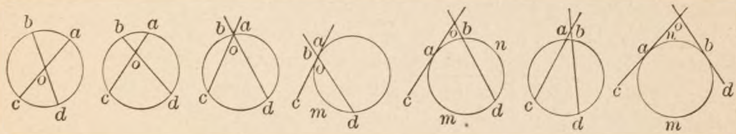
Thus, the angle ACB described by a line rotating about C from CA to CB is positive, and the arc AB is positive; the angle ACB' described by the line rotating about C from CA to CB' is negative, and the arc AB' is negative.

298. The Principle of Continuity. By marking the distinction between quantities measured in opposite directions, a theorem may often be so stated as to include two or more particular theorems.

The following theorem furnishes a good illustration:

299. *The angle included between two lines of unlimited length that cut or touch a circumference is measured by half the sum of the intercepted arcs.*

Here the word *sum* means the algebraic sum and includes both the arithmetical sum and the arithmetical difference of two quantities.



1. If the lines intersect at the centre, the two intercepted arcs are equal, and half the sum will be one of the arcs (§ 288).

2. If the lines intersect between the centre and the circumference, the angle is measured by half the sum of the arcs (§ 294).

3. If the lines intersect on the circumference, one of the arcs becomes zero and we have an inscribed angle (§ 289), or an angle formed by a tangent and a chord (§ 295). In each case the angle is measured by half the sum of the intercepted arcs.

4. If the lines intersect without the circumference, then the arc *ab* is negative and the algebraic sum is the arithmetical difference of the included arcs.

When the reasoning employed to prove a theorem is continued in the manner just illustrated to include two or more theorems, we are said to reason by the *Principle of Continuity.*

REVIEW QUESTIONS ON BOOK II.

1. What do we call the locus of points in a plane that are equidistant from a fixed point in the plane?

2. What does the chord of a segment become when the segment is a semicircle?

3. What kind of an angle do the radii of a sector include when the sector is a semicircle?

4. What is the difference between a chord and a secant?

5. What part of a tangent is meant by a tangent to a circle from an external point?

6. Two chords are equal in equal circles under either of two conditions. What are the two conditions?

7. Points that lie in a straight line are called *collinear;* points that lie in a circumference are called *concyclic.* How many collinear points can be concyclic?

8. What is meant by the statement that a central angle is measured by the arc intercepted between its sides?

9. What is an inscribed angle? What is its measure?

10. What kind of an angle is the angle inscribed in a semicircle? in a segment less than a semicircle? in a segment greater than a semicircle?

11. What is the measure of an angle included by two intersecting chords? by two secants intersecting without the circle?

12. What is the measure of an angle included by a tangent and a chord drawn to the point of contact?

13. When are two quantities of the same kind incommensurable?

14. When are two quantities of the same kind commensurable?

15. Define a variable and the limit of a variable.

16. Does the series $\frac{1}{2}$ in., $\frac{3}{4}$ in., $\frac{7}{8}$ in., $\frac{15}{16}$ in., etc., constitute a variable? Is the variable increasing or decreasing?

17. What is the limit of this variable?

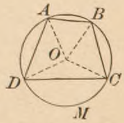
18. What is the test of a limit?

THEOREMS.

Ex. 101. An angle formed by a tangent and a chord is equal to the angle inscribed in the opposite segment.

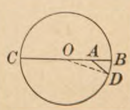
Ex. 102. Two chords drawn perpendicular to a third chord at its extremities are equal.

Ex. 103. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.



Ex. 104. If the sum of two opposite angles of a quadrilateral is equal to two right angles, a circle may be circumscribed about the quadrilateral.

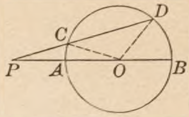
Let $\angle A + \angle C = 2$ rt. $\angle s$. Pass a circumference through D, A, and B, and prove that this circumference passes through C.



Ex. 105. The shortest line that can be drawn from a point within a circle to the circumference is the shorter segment of the diameter through that point.

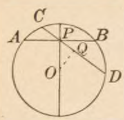
Let A be the given point. Prove AB shorter than any other line AD from A to the circumference.

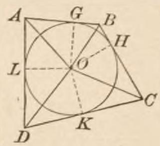
Ex. 106. The longest line that can be drawn from a point within a circle to the circumference is the longer segment of the diameter through that point.

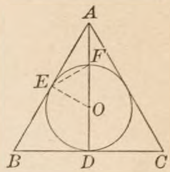


Ex. 107. The shortest line that can be drawn from a point without a circle to the circumference will pass through the centre of the circle if produced.

Ex. 108. The longest line that can be drawn from a point without a circle to the concave arc of the circumference passes through the centre of the circle.







Ex. 109. The shortest chord that can be drawn through a point within a circle is perpendicular to the diameter at that point.

Ex. 110. If two intersecting chords make equal angles with the diameter drawn through the point of intersection, the two chords are equal.

Rt. $\triangle\, COM =$ rt. $\triangle\, CON$. $\qquad \therefore\, OM = ON$.

Ex. 111. The angles subtended at the centre of a circle by any two opposite sides of a circumscribed quadrilateral are supplementary.

Ex. 112. The radius of a circle inscribed in an equilateral triangle is equal to one third the altitude of the triangle.

$\triangle\, OEF$ is equiangular and equilateral; $\angle FEA = \angle FAE$.

$\therefore\, AF = EF$. $\qquad \therefore\, AF = FO = OD$.

Ex. 113. The radius of a circle circumscribed about an equilateral triangle is equal to two thirds the altitude of the triangle (Ex. 27).

Ex. 114. A parallelogram inscribed in a circle is a rectangle.

Ex. 115. A trapezoid inscribed in a circle is an isosceles trapezoid.

Ex. 116. All chords of a circle which touch an interior concentric circle are equal, and are bisected at the point of contact.

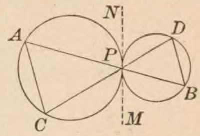
Ex. 117. If the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral (Ex. 116).

Ex. 118. If two circles are tangent to each other the tangents to them from any point of the common internal tangent are equal.

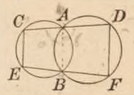
Ex. 119. If two circles touch each other and a line is drawn through the point of contact terminated by the circumferences, the tangents at its ends are parallel.

Ex. 120. If two circles touch each other and two lines are drawn through the point of contact terminated by the circumferences, the chords joining the ends of these lines are parallel.

$\angle A = \angle MPC$ and $\angle B = \angle NPD$. $\therefore \angle A = \angle B$.

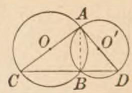


Ex. 121. If two circles intersect and a line is drawn through each point of intersection terminated by the circumferences, the chords joining the ends of these lines are parallel.



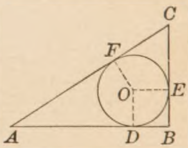
Ex. 122. Through one of the points of intersection of two circles a diameter of each circle is drawn. Prove that the line joining the ends of the diameters passes through the other point of intersection.

$$\angle ABC = \angle ABD = 90° \text{ (§ 290).}$$



Ex. 123. If two common external tangents or two common internal tangents are drawn to two circles, the segments intercepted between the points of contact are equal.

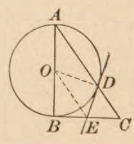
Ex. 124. The diameter of the circle inscribed in a right triangle is equal to the difference between the sum of the legs and the hypotenuse.



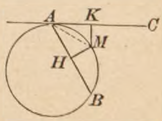
Ex. 125. If one leg of a right triangle is the diameter of a circle, the tangent at the point where the circumference cuts the hypotenuse bisects the other leg.

$\angle BOE = \angle DOE$. $\angle BOE = \angle OAD$.

$\therefore$ OE and AC are ∥. $\therefore BE = EC$ (§ 188).



Ex. 126. If, from any point in the circumference of a circle, a chord and a tangent are drawn, the perpendiculars dropped on them from the middle point of the subtended arc are equal. $\angle BAM = \angle CAM$.

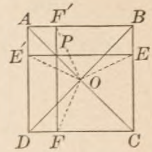


Ex. 127. The median of a trapezoid circumscribed about a circle is equal to one fourth the perimeter of the trapezoid (Ex. 103).

Ex. 128. Two fixed circles touch each other externally and a circle of variable radius touches both externally. Show that the difference of the distances from the centre of the variable circle to the centres of the fixed circles is constant.

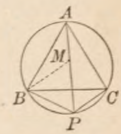
Ex. 129. If two fixed circles intersect, and circles are drawn to touch both, show that either the sum or the difference of the distances of their centres from the centres of the fixed circles is constant, according as they touch (i) one internally and one externally, (ii) both internally or both externally.

Ex. 130. If two straight lines are drawn through any point in a diagonal of a square parallel to the sides of the square, the points where these lines meet the sides lie on the circumference of a circle whose centre is the point of intersection of the diagonals.



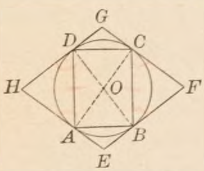
$\triangle POE = \triangle POF$ (§ 143). $\therefore OE = OF$. $\triangle POE' = \triangle POF'$.

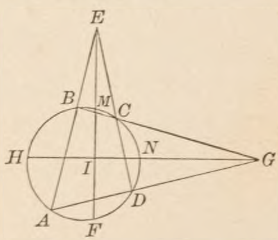
Ex. 131. If ABC is an inscribed equilateral triangle and P is any point in the arc BC, then $PA = PB + PC$.



Take $PM = PB$. $\triangle ABM = \triangle CBP$ (§ 143) and $AM = PC$.

Ex. 132. The tangents drawn through the vertices of an inscribed rectangle, which is not a square, enclose a rhombus.





Ex. 133. The bisectors of the angles included by the opposite sides (produced) of an inscribed quadrilateral intersect at right angles.

$$\text{Arc } AF - \text{arc } BM = \text{arc } DF - \text{arc } CM$$
$$\text{and arc } AH - \text{arc } DN = \text{arc } BH - \text{arc } CN.$$
$$\therefore \text{arc } FH + \text{arc } MN = \text{arc } HM + \text{arc } FN.$$
$$\therefore \angle FIH = \angle HIM.$$

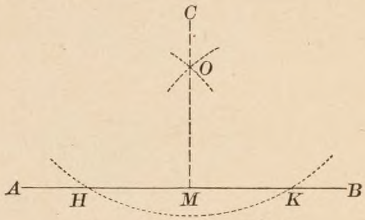
Discussion. This problem is impossible, if any two sides of the quadrilateral are parallel.

PROBLEMS OF CONSTRUCTION.

NOTE. Hitherto we have supposed the figures constructed. We now proceed to explain the methods of constructing simple problems, and afterwards to apply these methods to the solution of more difficult problems.

PROPOSITION XXI. PROBLEM.

300. *To let fall a perpendicular upon a given line from a given external point.*



Let AB be the given straight line, and C the given external point.

To let fall a $\perp$ *to the line* AB *from the point* C.

From C as a centre, with a radius sufficiently great, describe an arc cutting AB in two points, H and K.

From H and K as centres, with equal radii greater than $\frac{1}{2}$ HK,

describe two arcs intersecting at O.

Draw CO,

and produce it to meet AB at M.

CM is the $\perp$ required.

Proof. Since C and O are two points each equidistant from H and K, they determine a $\perp$ to HK at its middle point. § 161

Q.E.F.

NOTE. *Given* lines of the figures are represented by full lines, *resulting* lines by long-dashed, and *auxiliary* lines by short-dashed lines.

PROPOSITION XXII. PROBLEM.

301. *At a given point in a straight line, to erect a perpendicular to that line.*

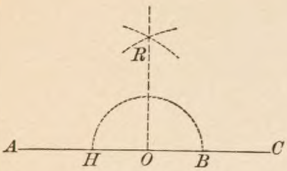


FIG. 1.

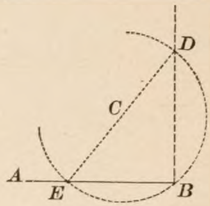


FIG. 2.

1. **Let O be the given point in AC.** Fig. 1.

Take OH and OB equal.

From H and B as centres, with equal radii greater than OB, describe two arcs intersecting at R. Join OR.

Then the line OR is the $\perp$ required.

Proof. O and R, two points each equidistant from H and B, determine the perpendicular bisector of HB. § 161

2. **Let B be the given point.** Fig. 2.

Take any point C without AB; and from C as a centre, with the distance CB as a radius, describe an arc intersecting AB at E.

Draw EC, and prolong it to meet the arc again at D.

Join BD, and BD is the $\perp$ required.

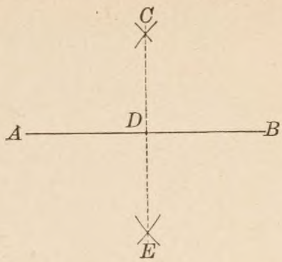
Proof. The $\angle B$ is a right angle. § 290

$\therefore BD$ is $\perp$ to AB. Q.E.F.

Discussion. The point C must be so taken that it will not be in the required perpendicular.

PROPOSITION XXIII. PROBLEM.

302. *To bisect a given straight line.*



To bisect the given straight line AB.

From A and B as centres, with equal radii greater than $\frac{1}{2}$ AB, describe arcs intersecting at C and E.

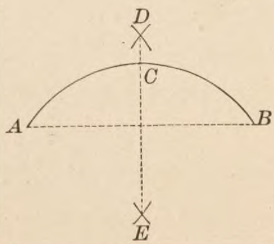
Join CE.

Then CE bisects AB. § 161

Q. E. F.

PROPOSITION XXIV. PROBLEM.

303. *To bisect a given arc.*



To bisect the given arc AB.

Draw the chord AB.

From A and B as centres, with equal radii greater than $\frac{1}{2}$ AB, describe arcs intersecting at D and E.

Draw DE.

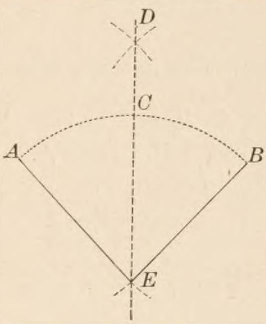
Then DE is the $\perp$ bisector of the chord AB. § 161

$\therefore DE$ bisects the arc ACB. § 248

Q.E.F.

PROPOSITION XXV. PROBLEM.

304. *To bisect a given angle.*



Let AEB be the given angle.

From E as a centre, with any radius, as EA, describe an arc cutting the sides of the $\angle E$ at A and B.

From A and B as centres, with equal radii greater than half the distance from A to B, describe two arcs intersecting at D.

Draw DE.

Then DE bisects the arc AB at C. § 303

$\therefore DE$ bisects the angle E. § 237

Q E. F.

Ex. 134. To construct an angle of 45°; of 135°.

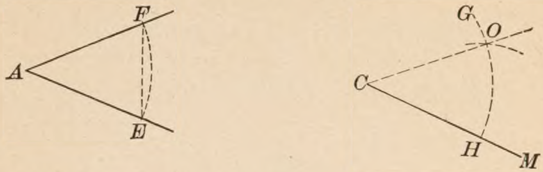
Ex. 135. To construct an equilateral triangle, having given one side.

Ex. 136. To construct an angle of 60°; of 150°.

Ex. 137. To trisect a right angle.

Proposition XXVI. Problem.

305. *At a given point in a given straight line, to construct an angle equal to a given angle.*



At C in the line CM, construct an angle equal to the given angle A.

From A as a centre, with any radius, AE, describe an arc cutting the sides of the $\angle A$ at E and F.

From C as a centre, with a radius equal to AE,
describe an arc HG cutting CM at H.

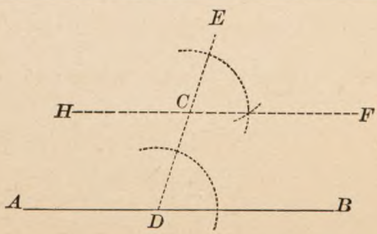
From H as a centre, with a radius equal to the chord EF,
describe an arc intersecting the arc HG at O.

Draw CO, and $\angle HCO$ is the required angle. Why?

Q. E. F.

Proposition XXVII. Problem.

306. *To draw a straight line parallel to a given straight line through a given external point.*



Let AB be the given line, and C the given point.

Draw ECD, making any convenient $\angle EDB$.

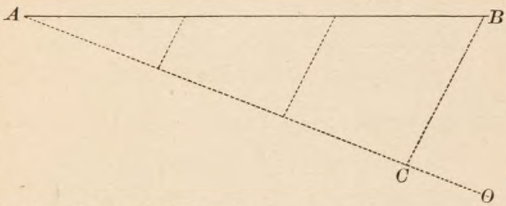
At the point C construct $\angle ECF$ equal to $\angle EDB$. § 305

Then the line HCF is $\parallel$ to AB. Why?

Q. E. F.

Proposition XXVIII. Problem.

307. *To divide a given straight line into a given number of equal parts.*



Let AB be the given straight line.

From A draw the line AO, making any convenient angle with AB.

Take any convenient length, and apply it to AO as many times as the line AB is to be divided into parts.

From C, the last point thus found on AO, draw CB.

Through the points of division on AO draw parallels to the line CB. § 306

These lines will divide AB into equal parts. § 187

Q. E. F.

Ex. 138. To construct an equilateral triangle, having given the perimeter.

Ex. 139. To divide a line into four equal parts by two different methods.

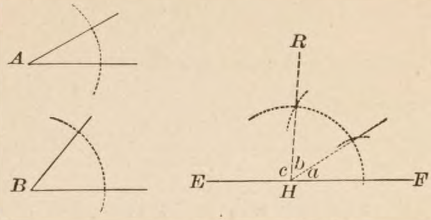
Ex. 140. Through a given point to draw a line which shall make equal angles with the two sides of a given angle.

Through the given point draw a $\perp$ to the bisector of the given $\angle$.

Ex. 141. To draw a line through a given point, so that it shall form with the sides of a given angle an isosceles triangle (Ex. 140).

PROPOSITION XXIX. PROBLEM.

308. *To find the third angle of a triangle when two of the angles are given.*



Let A and B be the two given angles.

At any point H in any line EF,

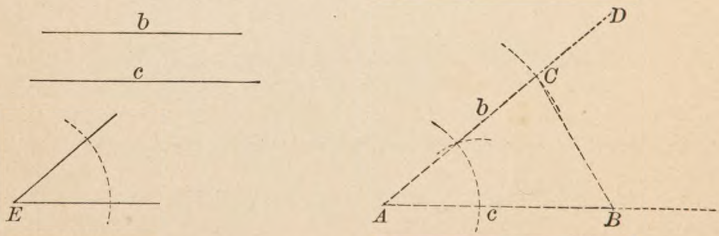
construct $\angle a$ equal to $\angle A$, and $\angle b$ equal to $\angle B$. § 305

Then $\angle c$ is the $\angle$ required. Why?

Q. E. F.

PROPOSITION XXX. PROBLEM.

309. *To construct a triangle when two sides and the included angle are given.*



Let b and c be the two sides of the triangle and E the included angle.

Take AB equal to the side c.

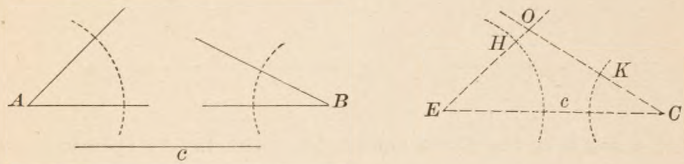
At A, construct $\angle BAD$ equal to the given $\angle E$. § 305

On AD take AC equal to b, and draw CB.

Then $\triangle ACB$ is the $\triangle$ required. Q.E.F.

PROPOSITION XXXI. PROBLEM.

310. *To construct a triangle when a side and two angles of the triangle are given.*



Let c be the given side, A and B the given angles.

Take EC equal to the side c.

At E construct the $\angle CEH$ equal to $\angle A$. § 305

At C construct the $\angle ECK$ equal to $\angle B$.

Produce EH and CK until they intersect at O.

Then $\triangle COE$ is the $\triangle$ required. Q.E.F.

REMARK. If one of the given angles is opposite to the given side, find the third angle by § 308, and proceed as above.

Discussion. The problem is impossible when the two given angles are together equal to or greater than two right angles.

Ex. 142. To construct an equilateral triangle, having given the altitude.

To construct an isosceles triangle, having given:

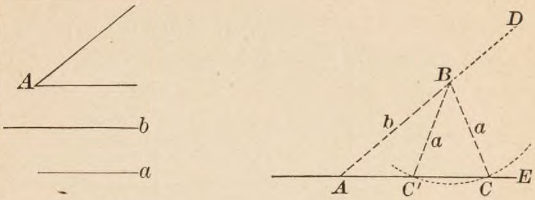
Ex. 143. The base and the altitude.

Ex. 144. The altitude and one of the legs.

Ex. 145. The angle at the vertex and the altitude.

Proposition XXXII. Problem.

311. *To construct a triangle when two sides and the angle opposite one of them are given.*



Let a and b be the given sides, and A the angle opposite a.

Case 1. *If a is less than b.*

Construct $\angle DAE$ equal to the given $\angle A$. § 305

On AD take AB equal to b.

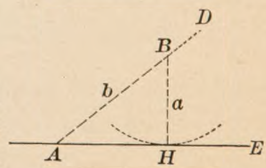
From B as a centre, with a radius equal to a,

describe an arc intersecting the line AE at C and C'.

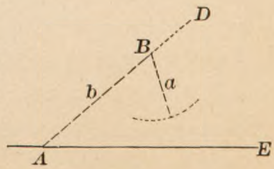
Draw BC and BC'.

Then both the $\triangle$ ABC and ABC' fulfil the conditions, and hence we have two constructions.

This is called the *ambiguous* case.



Discussion. If the side a is equal to the $\perp$ BH, the arc described from B will touch AE, and there will be but one construction, the right $\triangle$ ABH.

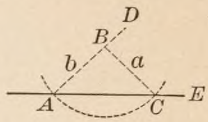


If the given side a is less than the $\perp$ from B, the arc described from B will not intersect or touch AE, and hence the problem is impossible.

If the $\angle A$ is right or obtuse, the problem is impossible; for the side opposite a right or obtuse angle is the greatest side. § 153

Case 2. *If a is equal to b.*

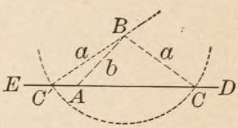
If the $\angle A$ is acute, and $a = b$, the arc described from B as a centre, and with a radius equal to a, will cut the line AE at the points A and C. There is therefore but one solution: the isosceles $\triangle ABC$.



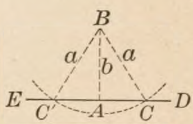
Discussion. If the $\angle A$ is right or obtuse, the problem is impossible; for equal sides of a $\triangle$ have equal $\angle$s opposite them, and a $\triangle$ cannot have two right $\angle$s or two obtuse $\angle$s.

Case 3. *If a is greater than b.*

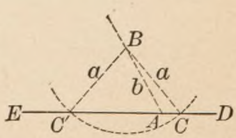
If the given $\angle A$ is acute, the arc described from B will cut the line ED on opposite sides of A, at C and C'. The $\triangle ABC$ answers the required conditions, but the $\triangle ABC'$ does not, for it does not contain the acute $\angle A$. There is then only one solution; namely, the $\triangle ABC$.



If the $\angle A$ is right, the arc described from B cuts the line ED on opposite sides of A, and we have two *equal* right $\triangle$s which fulfil the required conditions.



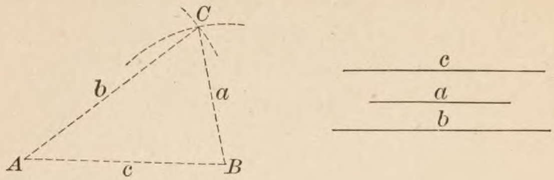
If the $\angle A$ is obtuse, the arc described from B cuts the line ED on opposite sides of A, at the points C and C'. The $\triangle ABC$ answers the required conditions, but the $\triangle ABC'$ does not, for it does not contain the obtuse $\angle A$. There is then only one solution; namely, the $\triangle ABC$.



Q.E.F.

Proposition XXXIII. Problem.

312. *To construct a triangle when the three sides of the triangle are given.*



Let the three sides be c, a, and b.

Take AB equal to c. From A as a centre, with a radius equal to b, describe an arc. From B as a centre, with a radius equal to a, describe an arc, intersecting the other arc at C.

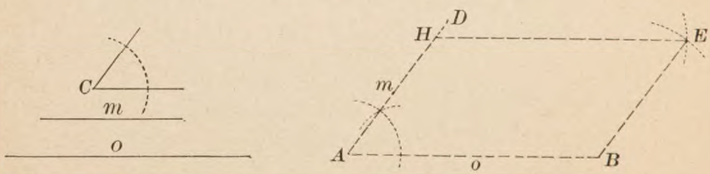
Draw CA and CB.

$\triangle$ CAB is the $\triangle$ required. Q. E. F.

Discussion. The problem is impossible when one side is equal to or greater than the sum of the other two sides.

Proposition XXXIV. Problem.

313. *To construct a parallelogram when two sides and the included angle are given.*



Let m and o be the two sides, and C the included angle.

Take AB equal to o.

At A construct $\angle$ BAD equal to $\angle$ C. § 305

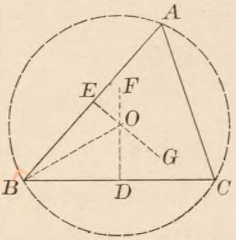
Take AH equal to m. From H as a centre, with a radius equal to o, describe an arc, and from B as a centre, with a radius equal to m, describe an arc, intersecting the other arc at E; and draw EH and EB.

The quadrilateral $ABEH$ is the ▱ required. § 182

Q. E. F.

Proposition XXXV. Problem.

314. *To circumscribe a circle about a given triangle.*



Let ABC be the given triangle.

Bisect AB and BC. § 302

At E and D, the points of bisection, erect ⊥s. § 301

Since BC is not the prolongation of AB, these ⊥s will intersect at some point O.

From O, with a radius equal to OB, describe a circle.

The ⊙ ABC is the ⊙ required.

Proof. The point O is equidistant from A and B,

and also is equidistant from B and C. § 160

∴ the point O is equidistant from A, B, and C,

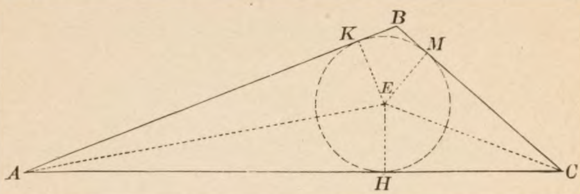
and a ⊙ described from O as a centre, with a radius equal to OB, will pass through the vertices A, B, and C. Q. E. F.

The same construction serves to describe a circumference which shall pass through three points not in the same straight line; also to find the centre of a given circle or of a given arc.

Note. The point O is called the *circum-centre* of the triangle.

Proposition XXXVI. Problem.

315. *To inscribe a circle in a given triangle.*



Let ABC be the given triangle.

Bisect the $\angle$s A and C. § 304

From E, the intersection of the bisectors,

draw $EH \perp$ to the side AC. § 300

From E as centre, with radius EH, describe the $\odot$ KHM.

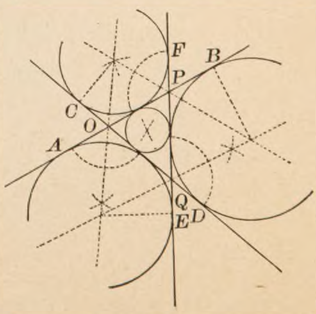
The $\odot$ KHM is the $\odot$ required.

Proof. Since E is in the bisector of the $\angle A$, it is equidistant from the sides AB and AC; and since E is in the bisector of the $\angle C$, it is equidistant from the sides AC and BC. § 162

$\therefore$ a $\odot$ described from E as centre, with a radius equal to EH, will touch the sides of the $\triangle$ and be inscribed in it. Q. E. F.

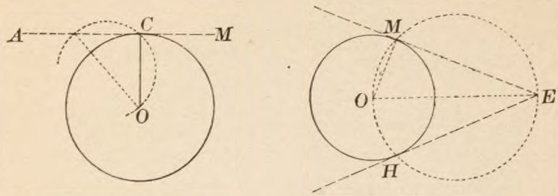
Note. The point E is called the *in-centre* of the triangle.

316. The intersections of the bisectors of the exterior angles of a triangle are the centres of three circles, each of which will touch one side of the triangle, and the two other sides produced. These three circles are called *escribed* circles; and their centres are called the *ex-centres* of the triangle.



PROPOSITION XXXVII. PROBLEM.

317. *Through a given point, to draw a tangent to a given circle.*



CASE 1. *When the given point is on the circumference.*

Let C be the given point on the circumference whose centre is O.

From the centre O draw the radius OC.

Through C draw $AM \perp$ to OC. § 301

Then AM is the tangent required. § 253

CASE 2. *When the given point is without the circle.*

Let O be the centre of the given circle, E the given point.

Draw OE.

On OE as a diameter, describe a circumference intersecting the given circumference at the points M and H.

Draw OM and EM.

Then EM is the tangent required.

Proof. $\angle OME$ is a right angle. § 290

$\therefore EM$ is tangent to the circle at M. § 253

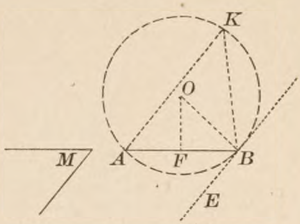
In like manner, we may prove EH tangent to the given $\odot$.

Q.E.F.

Ex. 146. To draw a tangent to a given circle, so that it shall be parallel to a given straight line.

Proposition XXXVIII. Problem.

318. *Upon a given straight line, to describe a segment of a circle in which a given angle may be inscribed.*



Let AB be the given line, and M the given angle.

Construct the $\angle ABE$ equal to the $\angle M$. § 305

Bisect the line AB by the $\perp$ OF. § 302

From the point B draw $BO \perp$ to EB. § 301

From O, the point of intersection of FO and BO, as a centre, with a radius equal to OB, describe a circumference.

The segment AKB is the segment required.

Proof. The point O is equidistant from A and B. § 160

$\therefore$ the circumference will pass through A.

But BE is $\perp$ to OB. Const.

$\therefore$ BE is tangent to the $\odot$, § 253

(*a straight line $\perp$ to a radius at its extremity is tangent to the* $\odot$).

$\therefore$ $\angle ABE$ is measured by $\frac{1}{2}$ arc AB, § 295

(*being an $\angle$ formed by a tangent and a chord*).

But any $\angle$ as $\angle K$ inscribed in the segment AKB is measured by $\frac{1}{2}$ arc AB. § 289

$\therefore$ the $\angle M$ may be inscribed in the segment AKB. Q.E.F.

SOLUTION OF PROBLEMS.

319. If a problem is so simple that the solution is obvious from a known theorem, we have only to make the construction according to the theorem, and then give a synthetic proof, if a proof is necessary, that the construction is correct, as in the examples of the fundamental problems already given.

320. But problems are usually of a more difficult type. The application of known theorems to their solution is not immediate, and often far from obvious. To discover the mode of application is the first and most difficult part of the solution. The best way to attack such problems is by a method resembling the analytic proof of a theorem, called the **analysis** of the problem.

1. **Suppose the construction made,** and let the figure represent all parts concerned, both given and required.

2. Study the relations among the parts with the aid of known theorems, and try to find some relation that will suggest the construction.

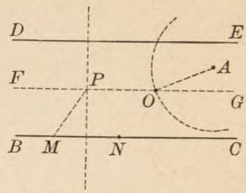
3. If this attempt fails, introduce new relations by drawing auxiliary lines, and study the new relations. If this attempt fails, make a new trial, and so on till a clue to the right construction is found.

321. A problem is *determinate* if it has a *definite* number of solutions, *indeterminate* if it has an *indefinite* number of solutions, and *impossible* if it has *no* solution. A problem is sometimes determinate for certain relative positions or magnitudes of the given parts, and indeterminate for other positions or magnitudes of the given parts.

322. The **discussion** of a problem consists in examining the problem with reference to all possible conditions, and in determining the conditions necessary for its solution.

Ex. 147. Problem. *To construct a circle that shall pass through a given point and cut chords of a given length from two parallels.*

Analysis. Suppose the problem solved. Let A be the given point, BC and DE the given parallels, MN the given length, and O the centre of the required circle.



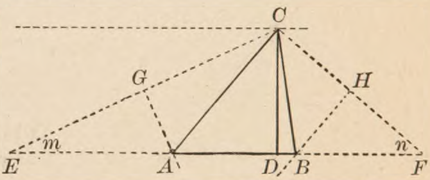
Since the circle cuts equal chords from two parallels its centre must be equidistant from them. Therefore, one locus for O is $FG \parallel$ to BC and equidistant from BC and DE.

Draw the $\perp$ bisector of MN, cutting FG in P. PM is the radius of the circle required. With A as centre and radius PM describe an arc cutting FG at O. Then O is the centre of the required circle.

Discussion. The problem is impossible if the distance from A to FG is greater than PM.

Ex. 148. Problem. *To construct a triangle, having given the perimeter, one angle, and the altitude from the vertex of the given angle.*

Analysis. Suppose the problem solved, and let ABC be the $\triangle$ required, ACB the given $\angle$, and CD the given altitude.



Produce AB both ways, and take $AE = AC$, and $BF = BC$, then EF = the given perimeter. Join CE and CF, forming the isosceles $\triangle$ CAE and CBF.

In the $\triangle ECF$, $\angle E + \angle F + \angle ECF = 180°$ (why ?), but $\angle ECF = \angle ECA + \angle FCB + \angle ACB$.

Since $\angle E = \angle ECA$ and $\angle F = \angle FCB$, we have $\angle ECF = \angle E + \angle F + \angle ACB$. $\therefore 2\angle E + 2\angle F + \angle ACB = 180°$.

$\therefore \angle E + \angle F + \frac{1}{2}\angle ACB = 90°$, and $\angle E + \angle F = 90° - \frac{1}{2}\angle ACB$.

By substitution, $\angle ECF = 90° + \frac{1}{2}\angle ACB$.

$\therefore \angle ECF$ is known.

Construction. To find the point C, construct on EF a segment that will contain the $\angle ECF$ (§ 318), and draw a parallel to EF at the distance CD, the given altitude.

To find the points A and B, draw the $\perp$ bisectors of the lines CE and CF, and the points A and B will be vertices of the required $\triangle$. Why ?

PROBLEMS OF CONSTRUCTION.

Ex. 149. Find the locus of a point at a given distance from a given circumference.

Find the locus of the centre of a circle:

Ex. 150. Which has a given radius r and passes through a given point P.

Ex. 151. Which has a given radius r and touches a given line AB.

Ex. 152. Which passes through two given points P and Q.

Ex. 153. Which touches a given straight line AB at a given point P.

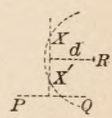
Ex. 154. Which touches each of two given parallels.

Ex. 155. Which touches each of two given intersecting lines.

Ex. 156. To find in a given line a point X which is equidistant from two given points.

The required point is the intersection of the given line with the perpendicular bisector of the line joining the two given points (§ 160).

Ex. 157. To find a point X equidistant from three given points.



Ex. 158. To find a point X equidistant from two given points and at a given distance from a third given point.

Ex. 159. To construct a circle which has a given radius and passes through two given points.

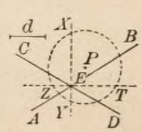
Ex. 160. To find a point X at given distances from two given points.

Ex. 161. To construct a circle which has its centre in a given line and passes through two given points.

Ex. 162. To find a point X equidistant from two given points and also equidistant from two given intersecting lines (§§ 160 and 162).

Ex. 163. To find a point X equidistant from two given points and also equidistant from two given parallel lines.

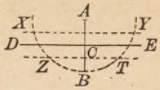
Ex. 164. To find a point X equidistant from two given intersecting lines and also equidistant from two given parallels.



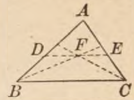
Ex. 165. To find a point X equidistant from two given intersecting lines and at a given distance from a given point.

Ex. 166. To find a point X which lies in one side of a given triangle and is equidistant from the other two sides.

Ex. 167. A straight railway passes two miles from a town. A place is four miles from the town and one mile from the railway. To find by construction the places that answer this description.

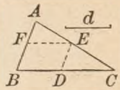


Ex. 168. In a triangle ABC, to draw DE parallel to the base BC, cutting the sides of the triangle in D and E, so that DE shall equal $DB + EC$ (§ 162).



Ex. 169. To draw through two sides of a triangle a line parallel to the third side so that the part intercepted between the sides shall have a given length.

Take $BD = d$.



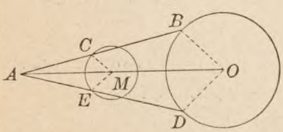
Ex. 170. Prove that the locus of the vertex of a right triangle, having a given hypotenuse as base, is the circumference described upon the given hypotenuse as diameter (§ 290).

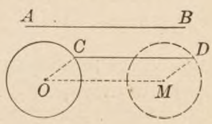
Ex. 171. Prove that the locus of the vertex of a triangle, having a given base and a given angle at the vertex, is the arc which forms with the base a segment capable of containing the given angle (§ 318).

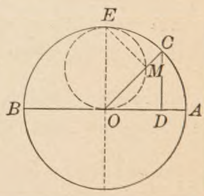
Ex. 172. Find the locus of the middle point of a chord of a given length that can be drawn in a given circle.

Ex. 173. Find the locus of the middle point of a chord drawn from a given point in a given circumference.

Ex. 174. Find the locus of the middle point of a straight line drawn from a given exterior point to a given circumference.



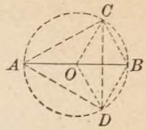




Ex. 175. A straight line moves so that it remains parallel to a given line, and touches at one end a given circumference. Find the locus of the other end.

Ex. 176. A straight rod moves so that its ends constantly touch two fixed rods which are perpendicular to each other. Find the locus of its middle point.

Ex. 177. In a given circle let AOB be a diameter, OC any radius, CD the perpendicular from C to AB. Upon OC take OM equal to CD. Find the locus of the point M as OC turns about O.



Ex. 178. To construct an equilateral triangle, having given the radius of the circumscribed circle.

To construct an isosceles triangle, having given:

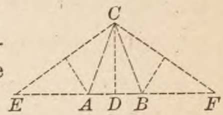
Ex. 179. The angle at the vertex and the base (§ 160 and § 318).

Ex. 180. The base and the radius of the circumscribed circle.

Ex. 181. The base and the radius of the inscribed circle.

Ex. 182. The perimeter and the altitude.

Let ABC be the △ required, EF the given perimeter. The altitude CD passes through the middle of EF, and the ⨺ AEC, BFC are isosceles.



To construct a right triangle, having given:

Ex. 183. The hypotenuse and one leg.

Ex. 184. One leg and the altitude upon the hypotenuse.

Ex. 185. The median and the altitude drawn from the vertex of the right angle.

Ex. 186. The hypotenuse and the altitude upon the hypotenuse.

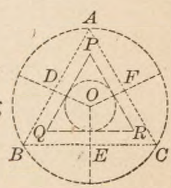
Ex. 187. The radius of the inscribed circle and one leg.

Ex. 188. The radius of the inscribed circle and an acute angle.

Ex. 189. An acute angle and the sum of the legs.

Ex. 190. An acute angle and the difference of the legs.

Ex. 191. To construct an equilateral triangle, having given the radius of the inscribed circle.



To construct a triangle, having given:

Ex. 192. The base, the altitude, and an angle at the base.

Ex. 193. The base, the altitude, and the ∠ at the vertex.

Ex. 194. The base, the corresponding median, and the ∠ at the vertex.

Ex. 195. The perimeter and the angles.

Ex. 196. One side, an adjacent ∠, and the sum of the other sides.

To construct a triangle, having given:

Ex. 197. One side, an adjacent $\angle$, and the difference of the other sides.

Ex. 198. The sum of two sides and the angles.

Ex. 199. One side, an adjacent $\angle$, and the radius of the circumscribed circle.

Ex. 200. The angles and the radius of the circumscribed circle.

Ex. 201. The angles and the radius of the inscribed circle.

Ex. 202. An angle, and the bisector and the altitude drawn from the vertex of the given angle.

Ex. 203. Two sides and the median corresponding to the other side.

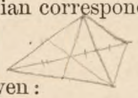
The required is the △ AF

AC = the on

AB = oth

BC = two tim

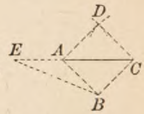
Ex. 204. The three medians.



To construct a square, having given:

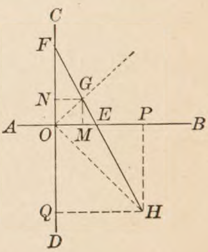
Ex. 205. The diagonal.

Ex. 206. The sum of the diagonal and one side.



Let $ABCD$ be the square required, CA the diagonal. Produce CA, making $AE = AB$. △ ABC and ABE are isosceles and $\angle BAC = \angle BCA = 45°$.

Ex. 207. Given two perpendiculars, AB and CD, intersecting in O, and a straight line intersecting these perpendiculars in E and F; to construct a square, one of whose angles shall coincide with one of the right angles at O, and the vertex of the opposite angle of the square shall lie in EF. (Two solutions.)



To construct a rectangle, having given:

Ex. 208. One side and the angle between the diagonals.

Ex. 209. The perimeter and the diagonal.

Ex. 210. The perimeter and the angle between the diagonals.

Ex. 211. The difference of two adjacent sides and the angle between the diagonals.

To construct a rhombus, having given:

Ex. 212. The two diagonals.

Ex. 213. One side and the radius of the inscribed circle.

Ex. 214. One angle and the radius of the inscribed circle.

Ex. 215. One angle and one of the diagonals.

To construct a rhomboid, having given:

Ex. 216. One side and the two diagonals.

Ex. 217. The diagonals and the angle between them.

Ex. 218. One side, one angle, and one diagonal.

Ex. 219. The base, the altitude, and one angle.

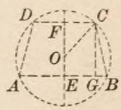
To construct an isosceles trapezoid, having given:

Ex. 220. The bases and one angle.

Ex. 221. The bases and the altitude.

Ex. 222. The bases and the diagonal.

Ex. 223. The bases and the radius of the circumscribed circle.



Let $ABCD$ be the isosceles trapezoid required, O the centre of the circumscribed $\odot$. A diameter $\perp$ to AB is $\perp$ to CD, and bisects both AB and CD. Draw $CG \parallel$ to FE. Then $EG = FC = \frac{1}{2} DC$.

To construct a trapezoid, having given:

Ex. 224. The four sides.

Ex. 225. The two bases and the two diagonals.

Ex. 226. The bases, one diagonal, and the $\angle$ between the diagonals.

To construct a circle which has the radius r and which also:

Ex. 227. Touches each of two intersecting lines AB and CD.

Ex. 228. Touches a given line AB and a given circle K.

Ex. 229. Passes through a given point P and touches a given line AB.

Ex. 230. Passes through a given point P and touches a given circle K.

To construct a circle which shall:

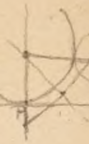
Ex. 231. Touch two given parallels and pass through a given point P.

Ex. 232. Touch three given lines two of which are parallel.

Ex. 233. Touch a given line AB at P and pass through a given point Q.

Ex. 234. Touch a given circle at P and pass through a given point Q.

Ex. 235. Touch two given lines and touch one of them at a given point P.



Ex. 236. Touch a given line and touch a given circle at a point P.

Ex. 237. Touch a given line AB at P and also touch a given circle.

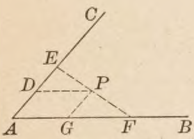
Ex. 238. To inscribe a circle in a given sector.

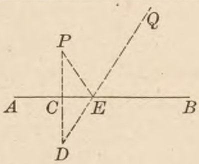
Ex. 239. To construct within a given circle three equal circles, so that each shall touch the other two and also the given circle.

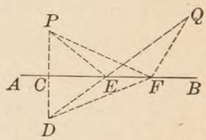
Ex. 240. To describe circles about the vertices of a given triangle as centres, so that each shall touch the two others.

Ex. 241. To bisect the angle formed by two lines, without producing the lines to their point of intersection.

Draw any line $EF \parallel$ to BA. Take $EG = EH$, and produce GH to meet BA at I. Draw the $\perp$ bisector of GI.







Ex. 242. To draw through a given point P between the sides of an angle BAC a line terminated by the sides of the angle and bisected at P.

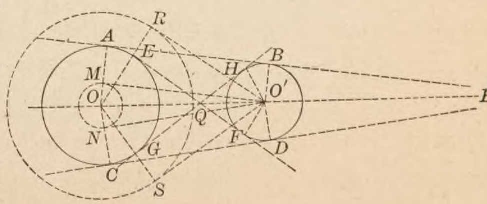
Ex. 243. Given two points P, Q, and a line AB; to draw lines from P and Q which shall meet on AB and make equal angles with AB.

Make use of the point which forms with P a pair of points symmetrical with respect to AB.

Ex. 244. To find the shortest path from P to Q which shall touch a line AB.

Ex. 245. To draw a common tangent to two given circles.

Let r and r' denote the radii of the circles, O and O' their centres. With centre O and radius $r - r'$ describe a $\odot$. From O' draw the tangents $O'M$, $O'N$. Produce OM and ON to meet the circumference at A and C. Draw the radii $O'B$ and $O'D \parallel$, respectively, to OA and OC. Draw AB and CD.



To draw the internal tangents use an auxiliary $\odot$ of radius $r + r'$.

Book III.

PROPORTION. SIMILAR POLYGONS.

THE THEORY OF PROPORTION.

323. A **proportion** is an expression of equality between two equal ratios; and is written in one of the following forms:

$$a : b = c : d; \quad a : b :: c : d; \quad \frac{a}{b} = \frac{c}{d}.$$

This proportion is read, "a is to b as c is to d"; or "the ratio of a to b is equal to the ratio of c to d."

324. The **terms** of a proportion are the four quantities compared; the *first* and *third* terms are called the **antecedents**, the *second* and *fourth* terms, the **consequents**; the *first* and *fourth* terms, the **extremes**, the *second* and *third* terms, the **means**.

Thus, in the proportion $a : b = c : d$; a and c are the antecedents, b and d the consequents, a and d the extremes, b and c the means.

325. The **fourth proportional** to three given quantities is the fourth term of the proportion which has for its first three terms the three given quantities *taken in order*.

Thus, d is the fourth proportional to a, b, and c in the proportion

$$a : b = c : d.$$

326. The quantities a, b, c, d, e, are said to be in **continued proportion**, if $a : b = b : c = c : d = d : e$.

If three quantities are in continued proportion, the second is called the **mean proportional** between the other two, and the third is called the **third proportional** to the other two.

Thus, in the proportion $a : b = b : c$; b is the mean proportional between a and c; and c is the third proportional to a and b.

Proposition I. Theorem.

327. *In every proportion the product of the extremes is equal to the product of the means.*

Let $a : b = c : d$.

Then $\frac{a}{b} = \frac{c}{d}$. § 323

Whence $ad = bc$. Q.E.D.

Proposition II. Theorem.

328. *The mean proportional between two quantities is equal to the square root of their product.*

Let $a : b = b : c$.

Then $b^2 = ac$. § 327

Whence, extracting the square root,

$$b = \sqrt{ac}.$$ Q.E.D.

Proposition III. Theorem.

329. *If the product of two quantities is equal to the product of two others, either two may be made the extremes of the proportion in which the other two are made the means.*

Let $ad = bc$.

To prove that $a : b = c : d$.

Divide both members of the given equation by bd.

Then $\frac{a}{b} = \frac{c}{d}$.

Or $a : b = c : d$. Q.E.D.

PROPOSITION IV. THEOREM.

330. *If four quantities are in proportion, they are in proportion by* **alternation**; *that is, the first term is to the third as the second is to the fourth.*

Let $\mathrm{a : b = c : d}$.

To prove that $a : c = b : d$.

Now $\dfrac{a}{b} = \dfrac{c}{d}$.

Multiply each member of the equation by $\dfrac{b}{c}$.

Then $\dfrac{a}{c} = \dfrac{b}{d}$.

Or $a : c = b : d$. Q.E.D.

PROPOSITION V. THEOREM.

331. *If four quantities are in proportion, they are in proportion by* **inversion**; *that is, the second term is to the first as the fourth is to the third.*

Let $\mathrm{a : b = c : d}$.

To prove that $b : a = d : c$.

Now $bc = ad$. § 327

Divide each member of the equation by ac.

Then $\dfrac{b}{a} = \dfrac{d}{c}$.

Or $b : a = d : c$. Q.E.D

Proposition VI. Theorem.

332. *If four quantities are in proportion, they are in proportion by* **composition**; *that is, the sum of the first two terms is to the second term as the sum of the last two terms is to the fourth term.*

Let $\mathbf{a : b = c : d}$.

To prove that $a + b : b = c + d : d$.

Now $$\frac{a}{b} = \frac{c}{d}.$$

Then $$\frac{a}{b} + 1 = \frac{c}{d} + 1;$$

that is, $$\frac{a+b}{b} = \frac{c+d}{d}.$$

Or $a + b : b = c + d : d$.

In like manner $a + b : a = c + d : c$. Q.E.D.

Proposition VII. Theorem.

333. *If four quantities are in proportion, they are in proportion by* **division**; *that is, the difference of the first two terms is to the second term as the difference of the last two terms is to the fourth term.*

Let $\mathbf{a : b = c : d}$.

To prove that $a - b : b = c - d : d$.

Now $$\frac{a}{b} = \frac{c}{d}.$$

Then $$\frac{a}{b} - 1 = \frac{c}{d} - 1;$$

that is, $$\frac{a-b}{b} = \frac{c-d}{d}.$$

Or $a - b : b = c - d : d$.

In like manner $a - b : a = c - d : c$. Q.E.D.

PROPOSITION VIII. THEOREM.

334. *If four quantities are in proportion, they are in proportion by* **composition and division**; *that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Let $a : b = c : d.$

Then $$\frac{a+b}{a} = \frac{c+d}{c}. \qquad \text{§ 332}$$

And $$\frac{a-b}{a} = \frac{c-d}{c}. \qquad \text{§ 333}$$

Divide, $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

Or $$a+b : a-b = c+d : c-d.$$ Q. E. D.

PROPOSITION IX. THEOREM.

335. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Let $a : b = c : d = e : f = g : h.$

To prove that $a + c + e + g : b + d + f + h = a : b.$

Let $$r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}.$$

Then $$a = br, \;\; c = dr, \;\; e = fr, \;\; g = hr.$$

And $$a + c + e + g = (b + d + f + h)\, r.$$

Divide by $(b + d + f + h)$.

Then $$\frac{a + c + e + g}{b + d + f + h} = r = \frac{a}{b}.$$

Or $$a + c + e + g : b + d + f + h = a : b.$$ Q. E. D.

Proposition X. Theorem.

336. *The products of the corresponding terms of two or more proportions are in proportion.*

Let $\quad a : b = c : d, \; e : f = g : h, \; k : l = m : n.$

To prove that $\quad aek : bfl = cgm : dhn.$

Now $\quad \frac{a}{b} = \frac{c}{d}, \; \frac{e}{f} = \frac{g}{h}, \; \frac{k}{l} = \frac{m}{n}.$

The products of the first members and of the second members of these equations give

$$\frac{aek}{bfl} = \frac{cgm}{dhn}.$$

Or $\quad aek : bfl = cgm : dhn.$ Q. E. D.

337. Cor. *If three quantities are in continued proportion, the first is to the third as the square of the first is to the square of the second.*

Proposition XI. Theorem.

338. *Like powers of the terms of a proportion are in proportion.*

Let $\quad a : b = c : d.$

To prove that $\quad a^n : b^n = c^n : d^n.$

Now $\quad \frac{a}{b} = \frac{c}{d}.$

Raise to the nth power, $\frac{a^n}{b^n} = \frac{c^n}{d^n}.$

Or $\quad a^n : b^n = c^n : d^n.$ Q. E. D.

339. Def. **Equimultiples** of two quantities are the products obtained by multiplying each of them by the same number. Thus, ma and mb are equimultiples of a and b.

PROPOSITION XII. THEOREM.

340. *Equimultiples of two quantities are in the same ratio as the quantities themselves.*

Let a and b be any two quantities.

To prove that $ma : mb = a : b.$

Now $$\frac{a}{b} = \frac{a}{b}.$$

Multiply both terms of the first fraction by m.

Then $$\frac{ma}{mb} = \frac{a}{b}.$$

Or $ma : mb = a : b.$ Q.E.D.

341. SCHOLIUM. In the treatment of proportion, it is assumed that the *quantities* involved are expressed by their *numerical measures.* It is evident that the ratio of two quantities of the same kind may be represented by a fraction, if the two quantities are expressed in *integers* in terms of a *common unit.* If there is no unit in terms of which both quantities can be expressed in *integers,* it is still possible by taking the unit of measure sufficiently small to find a fraction that will represent the ratio *to any required degree of accuracy.* § 269

If we speak of the product of two quantities, it must be understood that we mean simply *the product of the numbers which represent them when they are expressed in terms of a common unit.*

In order that four quantities, a, b, c, d, may form a proportion, a and b must be quantities of the same kind; and c and d must be quantities of the same kind; though c and d need not be of the same kind as a and b. In **alternation**, however, *the four quantities must be of the same kind.*

Proposition XIII. Theorem.

342. *If a line is drawn through two sides of a triangle parallel to the third side, it divides those sides proportionally.*

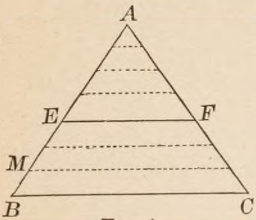


Fig. 1.

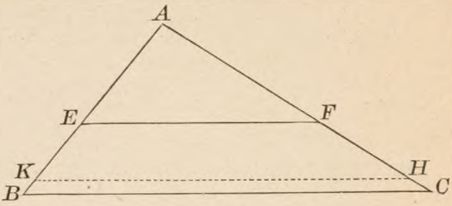


Fig. 2.

In the triangle ABC, let EF be drawn parallel to BC.

To prove that $EB : AE = FC : AF.$

Case 1. *When AE and EB* (Fig. 1) *are commensurable.*

Proof. Find a common measure of AE and EB, as MB.

Let MB be contained m times in EB, and n times in AE.

Then $EB : AE = m : n.$

At the points of division on BE and AE draw lines ∥ to BC. These lines will divide AC into $m + n$ equal parts, of which FC will contain m, and AF will contain n. § 187

$$\therefore FC : AF = m : n.$$

$$\therefore EB : AE = FC : AF. \qquad \text{Ax. 1}$$

Case 2. *When AE and EB* (Fig. 2) *are incommensurable.*

Proof. Divide AE into any number of equal parts, and apply one of these parts to EB as many times as EB will contain it.

Since AE and EB are incommensurable, a certain number of these parts will extend from E to some point K, leaving a remainder KB less than one of these parts. Draw KH ∥ to BC.

Then $EK : AE = FH : AF.$ Case 1

By increasing the *number* of equal parts into which AE is divided, we can make the *length* of each part less than any assigned value, however small, but not zero.

Hence, KB, which is less than one of these equal parts, has zero for a limit. § 275

And the corresponding segment HC has zero for a limit.

Therefore, EK approaches EB as a limit, § 271

and FH approaches FC as a limit.

Therefore, $\frac{EK}{AE}$ approaches $\frac{EB}{AE}$ as a limit, § 280

and $\frac{FH}{AF}$ approaches $\frac{FC}{AF}$ as a limit.

But $\frac{EK}{AE}$ is constantly equal to $\frac{FH}{AF}$. Case 1

$$\therefore \frac{EB}{AE} = \frac{FC}{AF}.$$ § 284

Q.E.D.

343. Cor. 1. *One side of a triangle is to either part cut off by a straight line parallel to the base as the other side is to the corresponding part.*

For $AE : EB = AF : FC.$

By composition, $AE + EB : AE = AF + FC : AF.$ § 332

Or $AB : AE = AC : AF.$

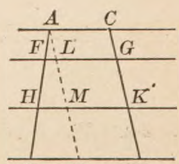
344. Cor. 2. *If two lines are cut by any number of parallels, the corresponding intercepts are proportional.*

Draw $AN \parallel$ to CD. Then

$AL = CG$, $LM = GK$, $MN = KD$. § 180

Now $AH : AM = AF : AL = FH : LM$

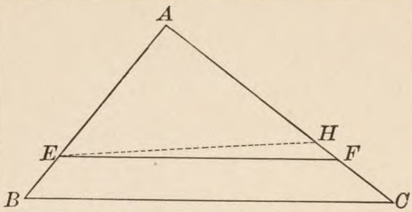
$= HB : MN.$ § 343



Or $AF : CG = FH : GK = HB : KD.$

PROPOSITION XIV. THEOREM.

345. *If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.*



In the triangle ABC, let EF be drawn so that

$$\frac{AB}{AE} = \frac{AC}{AF}.$$

To prove that EF *is* $\parallel$ *to* BC.

Proof. From E draw $EH \parallel$ to BC.

Then $AB : AE = AC : AH$, § 342

(*if a line is drawn through two sides of a* $\triangle$ *parallel to the third side, it divides these sides proportionally*).

But $AB : AE = AC : AF$. Hyp.

$\therefore AC : AF = AC : AH$. Ax. 1

$\therefore AF = AH$.

$\therefore EF$ and EH coincide. § 47

But EH is $\parallel$ to BC. Const.

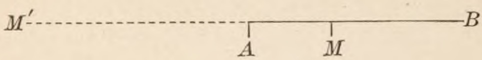
$\therefore EF$, which coincides with EH, is $\parallel$ to BC. Q.E.D.

Ex. 246. Find the fourth proportional to 91, 65, and 133.

Ex. 247. Find the mean proportional between 39 and 351.

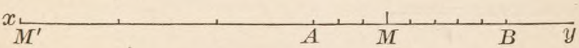
Ex. 248. Find the third proportional to 54 and 3.

346. If a given line AB is divided at M, a point between the extremities A and B, it is said to be divided **internally** into the segments MA and MB; and if it is divided at M', a point in the prolongation of AB, it is said to be divided **externally** into the segments $M'A$ and $M'B$.



In either case the segments are the *distances* from the point of division to the extremities of the line. If the line is divided internally, the *sum* of the segments is equal to the line; and if the line is divided externally, the *difference* of the segments is equal to the line.

Suppose it is required to divide the given line AB **internally and externally in the same ratio**; as, for example, the ratio of the two numbers 3 and 5.



We divide AB into $5 + 3$, or 8, equal parts, and take 3 parts from A; we then have the point M, such that

$$MA : MB = 3 : 5. \tag{1}$$

Secondly, we divide AB into $5 - 3$, or 2, equal parts, and lay off on the prolongation of AB, to the left of A, three of these equal parts; we then have the point M', such that

$$M'A : M'B = 3 : 5. \tag{2}$$

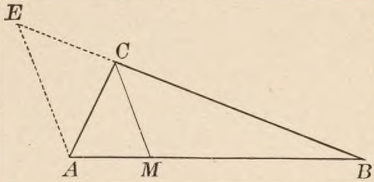
Comparing (1) and (2),

$$MA : MB = M'A : M'B.$$

347. Def. If a given straight line is divided internally and externally into segments having the same ratio, the line is said to be **divided harmonically.**

PROPOSITION XV. THEOREM.

348. *The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.*



Let CM bisect the angle C of the triangle CAB.

To prove that $MA : MB = CA : CB.$

Proof. Draw $AE \parallel$ to MC, meeting BC produced at E.

Then $MA : MB = CE : CB$, § 342

(*if a line is drawn through two sides of a* $\triangle$ *parallel to the third side, it divides those sides proportionally*).

Also, $\angle ACM = \angle CAE$, § 110

(*being alt.-int.* $\angle$s *of* $\parallel$ *lines*);

and $\angle BCM = \angle CEA$, § 112

(*being ext.-int.* $\angle$s *of* $\parallel$ *lines*).

But $\angle ACM = \angle BCM$. Hyp.

$\therefore \angle CAE = \angle CEA$. Ax. 1

$\therefore CE = CA$. § 147

Put CA for its equal, CE, in the first proportion.

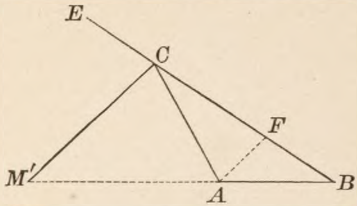
Then $MA : MB = CA : CB$. Q. E. D.

Ex. 249. In a triangle ABC, $AB = 12$, $AC = 14$, $BC = 13$. Find the segments of BC made by the bisector of the angle A.

Ex. 250. In a triangle CAB, $CA = 6$, $CB = 12$, $AB = 15$. Find the segments of AB made by the bisector of the angle C.

Proposition XVI. Theorem.

349. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments which are proportional to the adjacent sides.*



Let CM′ bisect the exterior angle ACE of the triangle CAB, and meet BA produced at M′.

To prove that $M'A : M'B = CA : CB.$

Proof. Draw $AF \parallel$ to $M'C$, meeting BC at F.

Then	$M'A : M'B = CF : CB.$	§ 343
Now	$\angle M'CE = \angle AFC,$	§ 112
and	$\angle M'CA = \angle CAF,$	§ 110
	(*being alt.-int.* $\angle$s *of* $\parallel$ *lines*).	
But	$\angle M'CE = \angle M'CA.$	Hyp.
	$\therefore \angle AFC = \angle CAF.$	Ax. 1
	$\therefore CA = CF.$	§ 147

Put CA for its equal, CF, in the first proportion.

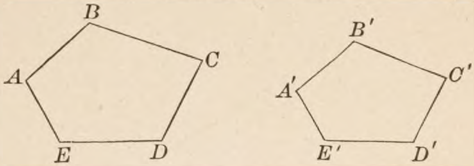
Then $M'A : M'B = CA : CB.$ Q.E.D.

Question. To what case does this theorem not apply? (See Ex. 41, page 69.)

350. Cor. *The bisectors of an interior angle and an exterior angle at one vertex of a triangle meeting the opposite side divide that side harmonically.* § 347

SIMILAR POLYGONS.

351. Def. **Similar polygons** are polygons that have their homologous angles equal, and their homologous sides proportional.



Thus, the polygons $ABCDE$ and $A'B'C'D'E'$ are similar, if the $\angle s\ A, B, C$, etc., are equal, respectively, to the $\angle s\ A', B', C'$, etc., and if

$$AB : A'B' = BC : B'C' = CD : C'D', \text{ etc.}$$

352. Def. **Homologous lines** in similar polygons are lines similarly situated.

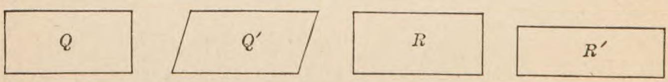
353. Def. The ratio of any two homologous lines in similar polygons, is called the **ratio of similitude** of the polygons.

The primary idea of similarity is **likeness of form.** The two conditions *necessary* to similarity are:

1. *For every angle in one of the figures there must be an equal angle in the other.*

2. *The homologous sides must be proportional.*

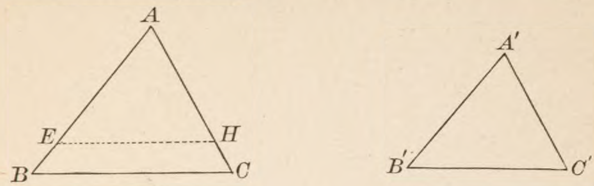
Thus, Q and Q' are not similar; the homologous sides are proportional, but the homologous angles are not equal. Also R and R' are not similar; the homologous angles are equal, but the sides are not proportional.



In the case of *triangles*, either condition involves the other (see § 354 and § 358).

Proposition XVII. Theorem.

354. *Two mutually equiangular triangles are similar.*



In the triangles ABC and A′B′C′, let the angles A, B, C be equal to the angles A′, B′, C′, respectively.

To prove that the $\triangle$ ABC and A′B′C′ are similar.

Since the $\triangle$ are mutually equiangular, we have only to prove that $AB : A'B' = AC : A'C' = BC : B'C'$. § 351

Proof. Place the $\triangle A'B'C'$ on the $\triangle ABC$ so that $\angle A'$ shall coincide with its equal, the $\angle A$; and $B'C'$ take the position EH.

Then $\angle AEH = \angle B$. Hyp.

$\therefore EH$ is $\parallel$ to BC. § 114

$\therefore AB : AE = AC : AH$. § 343

That is, $AB : A'B' = AC : A'C'$.

Similarly, by placing $\triangle A'B'C'$ on $\triangle ABC$, so that $\angle B'$ shall coincide with its equal, the $\angle B$, we may prove that

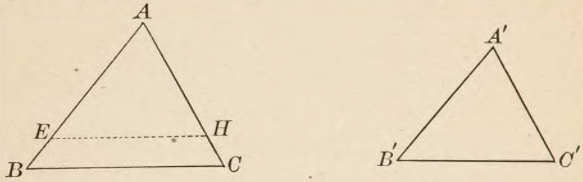
$$AB : A'B' = BC : B'C'.$$ Q.E.D.

355. Cor. 1. *Two triangles are similar if two angles of the one are equal, respectively, to two angles of the other.*

356. Cor. 2. *Two right triangles are similar if an acute angle of the one is equal to an acute angle of the other.*

Proposition XVIII. Theorem.

357. *If two triangles have an angle of the one equal to an angle of the other, and the including sides proportional, they are similar.*



In the triangles ABC and A′B′C′, let $\angle A = \angle A'$, and let

$$AB : A'B' = AC : A'C'.$$

To prove that the $\triangle$ *ABC and A'B'C' are similar.*

In this case we prove the $\triangle$ similar by proving them mutually equiangular.

Proof. Place the $\triangle A'B'C'$ on the $\triangle ABC$, so that the $\angle A'$ shall coincide with its equal, the $\angle A$.

Then the $\triangle A'B'C'$ will take the position of $\triangle AEH$.

Now $$\frac{AB}{A'B'} = \frac{AC}{A'C'}.$$ Hyp.

That is, $$\frac{AB}{AE} = \frac{AC}{AH}.$$

$\therefore EH$ is $\parallel$ to BC, § 345

(*if a line divides two sides of a $\triangle$ proportionally, it is $\parallel$ to the third side*).

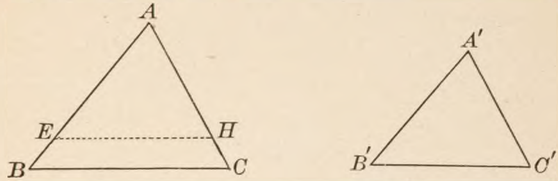
$\therefore \angle AEH = \angle B$, and $\angle AHE = \angle C$. § 112

$\therefore \triangle AEH$ is similar to $\triangle ABC$. § 354

$\therefore \triangle A'B'C'$ is similar to $\triangle ABC$. Q.E.D.

PROPOSITION XIX. THEOREM.

358. *If two triangles have their sides respectively proportional, they are similar.*



In the triangles ABC and A′B′C′, let

$$AB : A'B' = AC : A'C' = BC : B'C'.$$

To prove that the $\triangle$ *ABC and A′B′C′ are similar.*

In this case we prove the $\triangle$ similar by proving them mutually equiangular.

Proof. Upon AB take AE equal to $A'B'$, and upon AC take AH equal to $A'C'$.

Draw EH.

Then $AB : AE = AC : AH$. Hyp.

$\therefore \triangle ABC$ and AEH are similar. § 357

$\therefore AB : AE = BC : EH$; § 351

that is, $AB : A'B' = BC : EH$.

But $AB : A'B' = BC : B'C'$. Hyp.

$\therefore BC : EH = BC : B'C'$. Ax. 1

$\therefore EH = B'C'$.

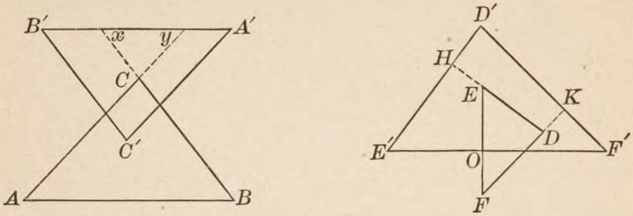
Hence, the $\triangle$ AEH and $A'B'C'$ are equal. § 150

But $\triangle AEH$ is similar to $\triangle ABC$.

$\therefore \triangle A'B'C'$ is similar to $\triangle ABC$. Q. E. D.

Proposition XX. Theorem.

359. *Two triangles which have their sides respectively parallel, or respectively perpendicular, are similar.*



Let ABC and A′B′C′ have their sides respectively parallel; and DEF and D′E′F′ have their sides respectively perpendicular.

To prove that the $\triangle$ *ABC and A′B′C′ are similar; and that the* $\triangle$ *DEF and D′E′F′ are similar.*

Proof. 1. Prolong BC and AC to $B'A'$, forming $\angle$s x and y.

Then $\angle B' = \angle x$ (§ 112), and $\angle B = \angle x$. § 110

Therefore, $\angle B' = \angle B$. Ax. 1

In like manner, $\angle A' = \angle A$.

Therefore, $\triangle A'B'C'$ is similar to $\triangle ABC$. § 355

2. Prolong DE and FD to meet $D'E'$ at H and $D'F'$ at K.

The quadrilateral $EHE'O$ has $\angle$s EHE' and $E'OE$ right angles, by hypothesis.

Therefore, $\angle E'$ and $\angle OEH$ are supplementary. § 206

But $\angle DEF$ and $\angle OEH$ are supplementary. § 86

Therefore, $\angle DEF = \angle E'$. § 85

In like manner, $\angle EDF = \angle D'$.

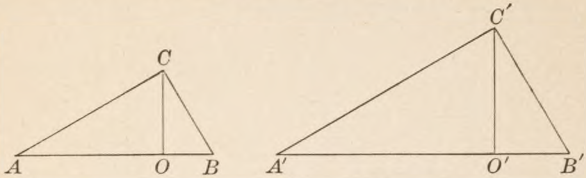
Therefore, $\triangle DEF$ is similar to $\triangle D'E'F'$. § 355

Q. E. D.

360. Cor. *The parallel sides and the perpendicular sides are homologous sides of the triangles.*

Proposition XXI. Theorem.

361. *The homologous altitudes of two similar triangles have the same ratio as any two homologous sides.*



In the two similar triangles ABC and A′B′C′, let CO and C′O′ be homologous altitudes.

To prove that $\frac{CO}{C'O'} = \frac{AC}{A'C'} = \frac{AB}{A'B'} = \frac{BC}{B'C'}$.

Proof. In the rt. △ COA and $C'O'A'$,

$$\angle A = \angle A', \qquad \text{§ 351}$$

(*being homologous* ∠ *of the similar* △ *ABC and A′B′C′*).

$$\therefore \triangle\, COA \text{ and } C'O'A' \text{ are similar}, \qquad \text{§ 356}$$

(*two rt.* △ *having an acute* ∠ *of the one equal to an acute* ∠ *of the other are similar*).

$$\therefore \frac{CO}{C'O'} = \frac{AC}{A'C'}. \qquad \text{§ 351}$$

In the similar △ ABC and $A'B'C'$,

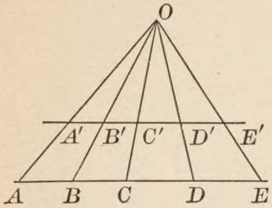
$$\frac{AC}{A'C'} = \frac{AB}{A'B'} = \frac{BC}{B'C'}. \qquad \text{§ 351}$$

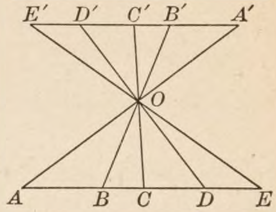
Therefore, $\frac{CO}{C'O'} = \frac{AC}{A'C'} = \frac{AB}{A'B'} = \frac{BC}{B'C'}$. Q.E.D.

Ex. 251. The base and altitude of a triangle are 7 feet 6 inches and 5 feet 6 inches, respectively. If the homologous base of a similar triangle is 5 feet 6 inches, find its homologous altitude.

PROPOSITION XXII. THEOREM.

362. *If two parallels are cut by three or more transversals that pass through the same point, the corresponding segments are proportional.*





Let the two parallels AE and A′E′ be cut by the transversals OA, OB, OC, OD, OE in A, A′, B, B′, etc.

To prove that $$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\frac{DE}{D'E'}.$$

Proof. Since $A'E'$ is $\parallel$ to AE, the pairs of $\triangle$ OAB and $OA'B'$, OBC and $OB'C'$, etc., are similar. § 354

$$\therefore \frac{AB}{A'B'}=\frac{OB}{OB'} \text{ and } \frac{BC}{B'C'}=\frac{OB}{OB'}, \qquad \text{§ 351}$$

(*homologous sides of similar* $\triangle$ *are proportional*).

$$\therefore \frac{AB}{A'B'}=\frac{BC}{B'C'}. \qquad \text{Ax. 1}$$

In a similar way it may be shown that

$$\frac{BC}{B'C'}=\frac{CD}{C'D'} \text{ and } \frac{CD}{C'D'}=\frac{DE}{D'E'}.$$

Q. E. D.

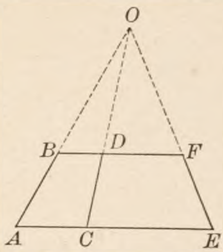
NOTE. A condensed form of writing the above is

$$\frac{AB}{A'B'}=\left(\frac{OB}{OB'}\right)=\frac{BC}{B'C'}=\left(\frac{OC}{OC'}\right)=\frac{CD}{C'D'}=\left(\frac{OD}{OD'}\right)=\frac{DE}{D'E'}.$$

A parenthesis about a ratio signifies that this ratio is used to prove the equality of the ratios immediately preceding and following it.

PROPOSITION XXIII. THEOREM.

363. CONVERSELY: *If three or more non-parallel straight lines intercept proportional segments upon two parallels, they pass through a common point.*



Let AB, CD, EF cut the parallels AE and BF so that

$$AC : BD = CE : DF.$$

To prove that AB, CD, EF prolonged meet in a point.

Proof. Prolong AB and CD until they meet in O.

Draw OE.

Designate by F' the point where OE cuts BF.

Then	$AC : BD = CE : DF'$.	§ 362
But	$AC : BD = CE : DF$.	Hyp.
	$\therefore CE : DF' = CE : DF$.	Ax. 1
	$\therefore DF' = DF$.	
	$\therefore F'$ coincides with F.	
	$\therefore EF$ coincides with EF'.	§ 47

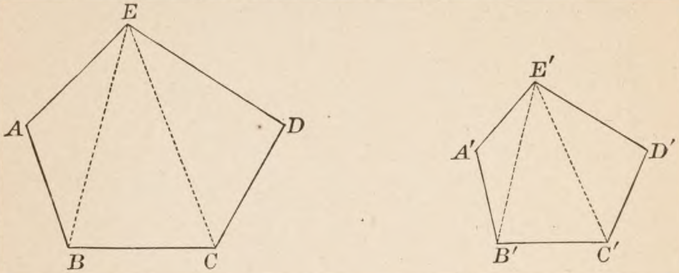
$\therefore EF$ prolonged passes through O.

$\therefore AB$, CD, and EF prolonged meet in the point O.

Q.E.D.

PROPOSITION XXIV. THEOREM.

364. *The perimeters of two similar polygons have the same ratio as any two homologous sides.*



Let the two similar polygons be ABCDE and A′B′C′D′E′, and let P and P′ represent their perimeters.

To prove that $P : P' = AB : A'B'$.

Proof. $AB : A'B' = BC : B'C' = CD : C'D'$, etc. § 351

$\therefore AB + BC +$ etc. $: A'B' + B'C' +$ etc. $= AB : A'B'$, § 335

(*in a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent*).

That is, $P : P' = AB : A'B'$. Q. E. D.

Ex. 252. If the line joining the middle points of the bases of a trapezoid is produced, and the two legs are also produced, the three lines will meet in the same point.

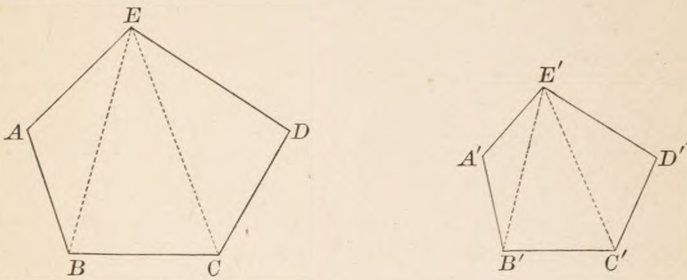
Ex. 253. AB and AC are chords drawn from any point A in the circumference of a circle, and AD is a diameter. The tangent to the circle at D intersects AB and AC at E and F, respectively. Show that the triangles ABC and AEF are similar.

Ex. 254. AD and BE are two altitudes of the triangle CAB. Show that the triangles CED and CAB are similar.

Ex. 255. If two circles are tangent to each other, the chords formed by a straight line drawn through the point of contact have the same ratio as the diameters of the circles.

PROPOSITION XXV. THEOREM.

365. *If two polygons are similar, they are composed of the same number of triangles, similar each to each, and similarly placed.*



Let the polygons ABCDE and A′B′C′D′E′ be similar.

From two homologous vertices, as E and E', draw diagonals EB, EC, and $E'B'$, $E'C'$.

To prove that the $\triangle$ *EAB, EBC, ECD are similar, respectively, to the* $\triangle$ *E′A′B′, E′B′C′, E′C′D′.*

Proof. The $\triangle$ EAB and $E'A'B'$ are similar. § 357

For $\angle A = \angle A'$, § 351

and $AE : A'E' = AB : A'B'$. § 351

Now $\angle ABC = \angle A'B'C'$, § 351

and $\angle ABE = \angle A'B'E'$. § 351

By subtracting, $\angle EBC = \angle E'B'C'$. Ax. 3

Now $EB : E'B' = AB : A'B'$, § 351

and $BC : B'C' = AB : A'B'$. § 351

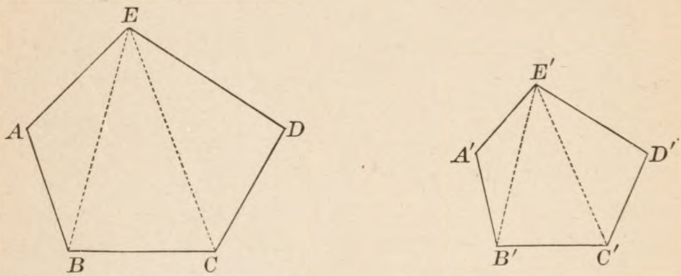
$\therefore EB : E'B' = BC : B'C'$. Ax. 1

$\therefore$ $\triangle$ EBC and $E'B'C'$ are similar. § 357

In like manner $\triangle$ ECD and $E'C'D'$ are similar. Q. E. D.

PROPOSITION XXVI. THEOREM.

366. CONVERSELY: *If two polygons are composed of the same number of triangles, similar each to each, and similarly placed, the polygons are similar.*



In the two polygons ABCDE and A′B′C′D′E′, let the triangles AEB, BEC, CED be similar, respectively, to the triangles A′E′B′, B′E′C′, C′E′D′; and similarly placed.

To prove that $ABCDE$ is similar to $A'B'C'D'E'$.

Proof. $\angle A = \angle A'$. § 351

Also, $\angle ABE = \angle A'B'E'$,

and $\angle EBC = \angle E'B'C'$. § 351

By adding, $\angle ABC = \angle A'B'C'$. Ax. 2

In like manner, $\angle BCD = \angle B'C'D'$, $\angle CDE = \angle C'D'E'$, etc.

Hence, the polygons are mutually equiangular.

Also, $\dfrac{AB}{A'B'} = \left(\dfrac{EB}{E'B'}\right) = \dfrac{BC}{B'C'} = \left(\dfrac{EC}{E'C'}\right) = \dfrac{CD}{C'D'}$, etc. § 351

Hence, the polygons have their homologous sides proportional.

Therefore, the polygons are similar. § 351

Q. E. D.

THEOREMS.

Ex. 256. If two circles are tangent to each other externally, the corresponding segments of two lines drawn through the point of contact and terminated by the circumferences are proportional.

Ex. 257. In a parallelogram $ABCD$, a line DE is drawn, meeting the diagonal AC in F, the side BC in G, and the side AB produced in E. Prove that $\overline{DF}^2 = FG \times FE$.

Ex. 258. Two altitudes of a triangle are inversely proportional to the corresponding bases.

Ex. 259. Two circles touch at P. Through P three lines are drawn, meeting one circle in A, B, C, and the other in A', B', C', respectively. Prove that the triangles ABC, $A'B'C'$ are similar.

Ex. 260. Two chords AB, CD intersect at M, and A is the middle point of the arc CD. Prove that the product $AB \times AM$ is constant if the chord AB is made to turn about the fixed point A.

Draw the diameter AE, and draw BE.

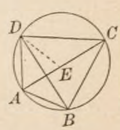
Ex. 261. If two circles touch each other, their common external tangent is the mean proportional between their diameters.

Let AB be the common tangent. Draw the diameters AC, BD. Join the point of contact P to A, B, C, and D. Show that APD and BPC are straight lines $\perp$ to each other, and that $\triangle CAB$, ABD are similar.

Ex. 262. If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionally by the circumference of the smaller circle.

Draw any two of the chords, and join the points where they meet the circumferences. The $\triangle$ thus formed are similar (Ex. 120).

Ex. 263. In an inscribed quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides.



Draw DE, making $\angle CDE = \angle ADB$. The $\triangle ABD$ and ECD are similar; and the $\triangle BCD$ and AED are similar.

Ex. 264. Two isosceles triangles with equal vertical angles are similar.

Ex. 265. The bisector of the vertical angle A of the triangle ABC intersects the base at D and the circumference of the circumscribed circle at E. Show that $AB \times AC = AD \times AE$.

NUMERICAL PROPERTIES OF FIGURES.

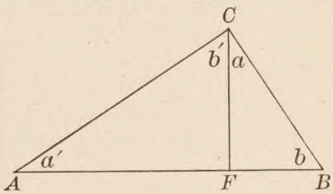
Proposition XXVII. Theorem.

367. *If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse:*

1. *The triangles thus formed are similar to the given triangle, and to each other.*

2. *The perpendicular is the mean proportional between the segments of the hypotenuse.*

3. *Each leg of the right triangle is the mean proportional between the hypotenuse and its adjacent segment.*



In the right triangle ABC, let CF be drawn from the vertex of the right angle C, perpendicular to AB.

1. *To prove that* $\triangle$ BCA, CFA, BFC *are similar.*

Proof. The rt. $\triangle$ CFA and BCA are similar, § 356

since the $\angle a'$ is common.

The rt. $\triangle$ BFC and BCA are similar, § 356

since the $\angle b$ is common.

Since the $\triangle$ CFA and BFC are each similar to $\triangle BCA$, they are similar to each other. § 354

2. *To prove that* $AF : CF = CF : FB.$

Proof. In the similar $\triangle$ CFA and BFC,

$$AF : CF = CF : FB. \qquad \text{§ 351}$$

3. *To prove that* $AB : AC = AC : AF,$

and $AB : BC = BC : BF.$

Proof. In the similar $\triangle$ BCA and CFA,

$$AB : AC = AC : AF. \quad \S\ 351$$

In the similar $\triangle$ BCA and BFC,

$$AB : BC = BC : BF. \quad \S\ 351$$

Q. E. D.

368. Cor. 1. *The squares of the two legs of a right triangle are proportional to the adjacent segments of the hypotenuse.*

From the proportions in § 367, 3,

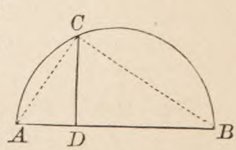
$$\overline{AC}^2 = AB \times AF, \text{ and } \overline{BC}^2 = AB \times BF. \quad \S\ 327$$

Hence, $$\frac{\overline{AC}^2}{\overline{BC}^2} = \frac{AB \times AF}{AB \times BF} = \frac{AF}{BF}.$$

369. Cor. 2. *The squares of the hypotenuse and either leg are proportional to the hypotenuse and the adjacent segment.*

For $$\frac{\overline{AB}^2}{\overline{AC}^2} = \frac{AB \times AB}{AB \times AF} = \frac{AB}{AF}.$$

370. Cor. 3. *The perpendicular from any point in the circumference to the diameter of a circle is the mean proportional between the segments of the diameter.*

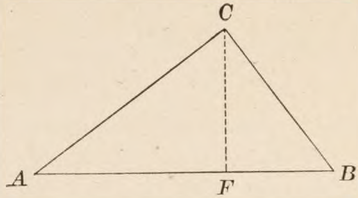


The chord drawn from any point in the circumference to either extremity of the diameter is the mean proportional between the diameter and the adjacent segment.

For the $\angle ACB$ is a rt. $\angle$. § 290

PROPOSITION XXVIII. THEOREM.

371. *The sum of the squares of the two legs of a right triangle is equal to the square of the hypotenuse.*



Let ABC be a right triangle with its right angle at C.

To prove that $\overline{AC}^2 + \overline{CB}^2 = \overline{AB}^2$.

Proof. Draw $CF \perp$ to AB.

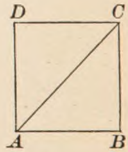
Then $\overline{AC}^2 = AB \times AF$,

and $\overline{CB}^2 = AB \times BF$. § 367

By adding, $\overline{AC}^2 + \overline{CB}^2 = AB(AF + BF) = \overline{AB}^2$. Q. E. D.

372. COR. 1. *The square of either leg of a right triangle is equal to the difference of the square of the hypotenuse and the square of the other leg.*

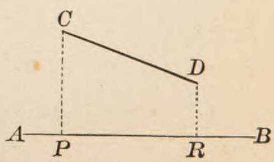
373. COR. 2. *The diagonal and a side of a square are incommensurable.*



For $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 = 2\overline{AB}^2$.

$\therefore AC = AB\sqrt{2}$.

374. DEF. The **projection** of any line upon a second line is the segment of the second line included between the perpendiculars drawn to it from the extremities of the first line. Thus, PR is the projection of CD upon AB.



PROPOSITION XXIX. THEOREM.

375. *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides by the projection of the other upon that side.*

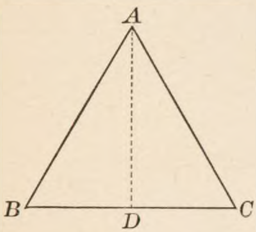


FIG. 1.

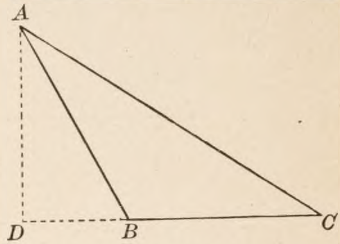


FIG. 2.

Let C be an acute angle of the triangle ABC, and DC the projection of AC upon BC.

To prove that $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times DC.$

Proof. If D falls upon the base (Fig. 1),

$$DB = BC - DC.$$

If D falls upon the base produced (Fig. 2),

$$DB = DC - BC.$$

In either case,

$$\overline{DB}^2 = \overline{BC}^2 + \overline{DC}^2 - 2\,BC \times DC.$$

Add $\overline{AD}^2$ to both sides of this equality, and we have

$$\overline{AD}^2 + \overline{DB}^2 = \overline{BC}^2 + \overline{AD}^2 + \overline{DC}^2 - 2\,BC \times DC.$$

But $\overline{AD}^2 + \overline{DB}^2 = \overline{AB}^2,$

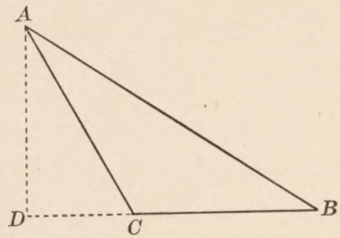
and $\overline{AD}^2 + \overline{DC}^2 = \overline{AC}^2.$ § 371

Put $\overline{AB}^2$ and $\overline{AC}^2$ for their equals in the above equality.

Then $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times DC.$ Q.E.D.

PROPOSITION XXX. THEOREM.

376. *In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of one of those sides by the projection of the other upon that side.*



Let C be the obtuse angle of the triangle ABC, and CD be the projection of AC upon BC produced.

To prove that $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\,BC \times DC.$

Proof. $DB = BC + DC.$

Squaring, $\overline{DB}^2 = \overline{BC}^2 + \overline{DC}^2 + 2\,BC \times DC.$

Add $\overline{AD}^2$ to both sides, and we have

$$\overline{AD}^2 + \overline{DB}^2 = \overline{BC}^2 + \overline{AD}^2 + \overline{DC}^2 + 2\,BC \times DC.$$

But $\overline{AD}^2 + \overline{DB}^2 = \overline{AB}^2$, and $\overline{AD}^2 + \overline{DC}^2 = \overline{AC}^2$. § 371

Put $\overline{AB}^2$ and $\overline{AC}^2$ for their equals in the above equality.

Then $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\,BC \times DC.$ Q.E.D.

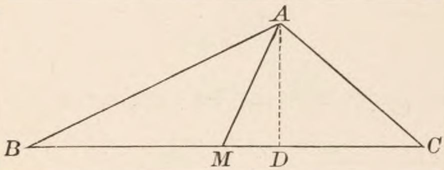
NOTE 1. By the Principle of Continuity the last three theorems may be included in one theorem. Let the student explain.

NOTE 2. The last three theorems enable us to compute the lengths of the altitudes of a triangle if the lengths of the three sides are known.

PROPOSITION XXXI. THEOREM.

377. 1. *The sum of the squares of two sides of a triangle is equal to twice the square of half the third side increased by twice the square of the median upon that side.*

2. *The difference of the squares of two sides of a triangle is equal to twice the product of the third side by the projection of the median upon that side.*



In the triangle ABC, let AM be the median and MD the projection of AM upon the side BC. Also, let AB be greater than AC.

To prove that 1. $\overline{AB}^2 + \overline{AC}^2 = 2\,\overline{BM}^2 + 2\,\overline{AM}^2.$

2. $\overline{AB}^2 - \overline{AC}^2 = 2\,BC \times MD.$

Proof. Since $AB > AC$, the $\angle AMB$ will be obtuse, and the $\angle AMC$ will be acute. § 155

Then $\overline{AB}^2 = \overline{BM}^2 + \overline{AM}^2 + 2\,BM \times MD,$ § 376

and $\overline{AC}^2 = \overline{MC}^2 + \overline{AM}^2 - 2\,MC \times MD.$ § 375

Add these two equalities, and observe that $BM = MC$.

Then $\overline{AB}^2 + \overline{AC}^2 = 2\,\overline{BM}^2 + 2\,\overline{AM}^2.$

Subtract the second equality from the first.

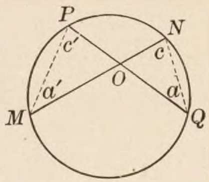
Then $\overline{AB}^2 - \overline{AC}^2 = 2\,BC \times MD.$ Q. E. D

NOTE. This theorem enables us to compute the lengths of the medians of a triangle if the lengths of the three sides are known.

because $\overline{AM}^2 = \frac{1}{2}(\overline{AB}^2 + \overline{AC}^2) - \overline{BM}^2$

PROPOSITION XXXII. THEOREM.

378. *If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.*



Let any two chords MN and PQ intersect at O.

To prove that $OM \times ON = OQ \times OP$.

Proof. Draw MP and NQ.

$\angle a = \angle a'$, § 289

(*each being measured by* $\frac{1}{2}$ *arc* PN).

And $\angle c = \angle c'$, § 289

(*each being measured by* $\frac{1}{2}$ *arc* MQ).

$\therefore$ the $\triangle$ NOQ and POM are similar. § 355

$\therefore OQ : OM = ON : OP$. § 351

$\therefore OM \times ON = OQ \times OP$. § 327

Q. E. D.

379. SCHOLIUM. This proportion may be written

$$\frac{OM}{OQ} = \frac{OP}{ON}, \text{ or } \frac{OM}{OQ} = \frac{1}{\frac{ON}{OP}};$$

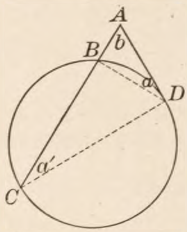
that is, the ratio of two corresponding segments is equal to the *reciprocal* of the ratio of the other two segments.

Hence, these segments are said to be *reciprocally proportional.*

380. Def. **A secant from a point to a circle** is understood to mean the segment of the secant lying between the point and the *second point* of intersection of the secant and circumference.

Proposition XXXIII. Theorem.

381. *If from a point without a circle a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.*



Let AD be a tangent and AC a secant drawn from the point A to the circle BCD.

To prove that $AC : AD = AD : AB$.

Proof. Draw DC and DB.

The △ ADC and ABD are similar. § 355

For $\angle b$ is common; and $\angle a' = \angle a$, §§ 289, 295

(each being measured by ½ arc BD).

$\therefore AC : AD = AD : AB.$ § 351

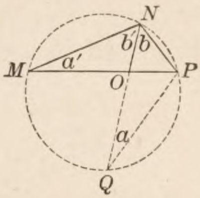
Q. E. D.

382. Cor. *If from a fixed point without a circle a secant is drawn, the product of the secant and its external segment is constant in whatever direction the secant is drawn.*

For $AC \times AB = \overline{AD}^2.$ § 327

PROPOSITION XXXIV. THEOREM.

383. *The square of the bisector of an angle of a triangle is equal to the product of the sides of this angle diminished by the product of the segments made by the bisector upon the third side of the triangle.*



Let NO bisect the angle MNP of the triangle MNP.

To prove that $\overline{NO}^2 = NM \times NP - OM \times OP$.

Proof. Circumscribe the $\odot\, MNP$ about the $\triangle\, MNP$. § 314

Produce NO to meet the circumference in Q, and draw PQ.

The $\triangle$s NQP and NMO are similar. § 355

For $\angle b = \angle b'$, Hyp.

and $\angle a = \angle a'$. § 289

Whence $NQ : NM = NP : NO$. § 351

$$\therefore NM \times NP = NQ \times NO$$
$$= (NO + OQ)\, NO$$
$$= \overline{NO}^2 + NO \times OQ.$$

But $NO \times OQ = MO \times OP$. § 378

$$\therefore MN \times NP = \overline{NO}^2 + MO \times OP.$$

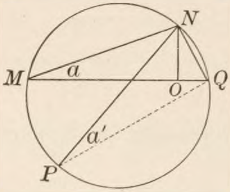
Whence $\overline{NO}^2 = NM \times NP - MO \times OP$. Ax. 3

Q. E. D.

NOTE. This theorem enables us to compute the lengths of the bisectors of the angles of a triangle if the lengths of the sides are known.

PROPOSITION XXXV. THEOREM.

384. *In any triangle the product of two sides is equal to the product of the diameter of the circumscribed circle by the altitude upon the third side.*



Let NMQ be a triangle, NO the altitude, and QNMP the circle circumscribed about the triangle NMQ.

Draw the diameter NP, and draw PQ.

To prove that $NM \times NQ = NP \times NO$.

Proof. In the $\triangle$ NOM and NQP,

$\angle NOM$ is a rt. $\angle$,	Hyp.
$\angle NQP$ is a rt. $\angle$,	§ 290
and $\angle a = \angle a'$,	§ 289
(*each being measured by* $\frac{1}{2}$ *arc* NQ).	
$\therefore$ $\triangle$ NOM and NQP are similar.	§ 356
Whence $NM : NP = NO : NQ$.	§ 351
$\therefore NM \times NQ = NP \times NO$.	§ 327

Q. E. D.

NOTE. This theorem enables us to compute the length of the radius of a circle circumscribed about a triangle, if the lengths of the three sides of the triangle are known.

Ex. 266. If OE, OF, OG are the perpendiculars from any point O within the triangle ABC upon the sides AB, BC, CA, respectively, show that $\overline{AE}^2 + \overline{BF}^2 + \overline{CG}^2 = \overline{EB}^2 + \overline{FC}^2 + \overline{GA}^2$.

THEOREMS.

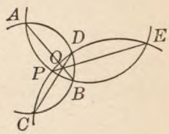
Ex. 267. The sum of the squares of the segments of two perpendicular chords is equal to the square of the diameter of the circle.

If AB, CD are the chords, draw the diameter BE, draw AC, ED, BD. Prove that $AC = ED$, and apply § 371.

Ex. 268. The tangents to two intersecting circles drawn from any point in their common chord produced, are equal. (§ 381.)

Ex. 269. The common chord of two intersecting circles, if produced, will bisect their common tangents. (§ 381.)

Ex. 270. If three circles intersect one another, the common chords all pass through the same point.



Let two of the chords AB and CD meet at O. Join the point of intersection E to O, and suppose that EO produced meets the same two circles at two different points P and Q. Then prove that $OP = OQ$ (§ 378); hence, that the points P and Q coincide.

Ex. 271. If two circles are tangent to each other, the common internal tangent bisects the two common external tangents.

Ex. 272. If the perpendiculars from the vertices of the triangle ABC upon the opposite sides intersect at D, show that

$$\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2.$$

Ex. 273. In an isosceles triangle, the square of a leg is equal to the square of any line drawn from the vertex to the base, increased by the product of the segments of the base.

Ex. 274. The squares of two chords drawn from the same point in a circumference have the same ratio as the projections of the chords on the diameter drawn from the same point.

Ex. 275. The difference of the squares of two sides of a triangle is equal to the difference of the squares of the segments of the third side, made by the perpendicular on the third side from the opposite vertex.

Ex. 276. E is the middle point of BC, one of the parallel sides of the trapezoid $ABCD$; AE and DE produced meet DC and AB produced at F and G, respectively. Show that FG is parallel to DA.

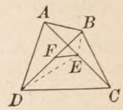
$\triangle$ AGD and BGE are similar; and $\triangle$ AFD and EFC are similar.

Ex. 277. If two tangents are drawn to a circle at the extremities of a diameter, the portion of a third tangent intercepted between them is divided at its point of contact into segments whose product is equal to the square of the radius.

Ex. 278. If two exterior angles of a triangle are bisected, the line drawn from the point of intersection of the bisectors to the opposite angle of the triangle bisects that angle.

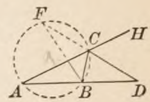
Ex. 279. The sum of the squares of the diagonals of a quadrilateral is equal to twice the sum of the squares of the lines that join the middle points of the opposite sides.

Ex. 280. The sum of the squares of the four sides of any quadrilateral is equal to the sum of the squares of the diagonals, increased by four times the square of the line joining the middle points of the diagonals.



Apply § 377 to the ⊿s ABC and ADC, add the results, and eliminate $\overline{BE}^2 + \overline{DE}^2$ by applying § 377 to the $\triangle\, BDE$.

Ex. 281. The square of the bisector of an exterior angle of a triangle is equal to the product of the external segments determined by the bisector upon one of the sides, diminished by the product of the other two sides.



Let CD bisect the exterior $\angle\, BCH$ of the $\triangle\, ABC$. ⊿s ACD and FCB are similar (§ 355). Apply § 382.

Ex. 282. If a point O is joined to the vertices of a triangle ABC; through any point A' in OA a line parallel to AB is drawn, meeting OB at B'; through B' a line parallel to BC, meeting OC at C'; and C' is joined to A'; the triangle $A'B'C'$ is similar to the triangle ABC.

Ex. 283. If the line of centres of two circles meets the circumferences at the consecutive points A, B, C, D, and meets the common external tangent at P, then $PA \times PD = PB \times PC$.

Ex. 284. The line of centres of two circles meets the common external tangent at P, and a secant is drawn from P, cutting the circles at the consecutive points E, F, G, H. Prove that $PE \times PH = PF \times PG$.

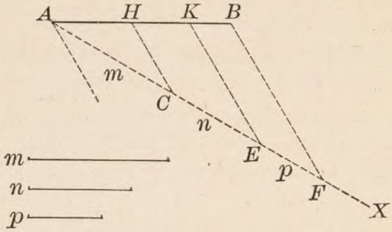
Draw radii to the points of contact, and to E, F, G, H. Let fall ⊥s on PH from the centres of the ⊙s. The various pairs of ⊿s are similar.

Ex. 285. If a line drawn from a vertex of a triangle divides the opposite side into segments proportional to the adjacent sides, the line bisects the angle at the vertex.

PROBLEMS OF CONSTRUCTION.

PROPOSITION XXXVI. PROBLEM.

385. *To divide a given straight line into parts proportional to any number of given lines.*



Let AB, m, n, and p be given straight lines.

To divide AB into parts proportional to m, n, and p.

Draw AX, making any convenient $\angle$ with AB.

On AX take AC equal to m, CE to n, EF to p.

Draw BF.

From E and C draw EK and $CH \parallel$ to FB.

Through A draw a line $\parallel$ to BF.

K and H are the division points required.

Proof. $$\frac{AH}{AC}=\frac{HK}{CE}=\frac{KB}{EF},$$ § 344

(*if two lines are cut by any number of parallels, the corresponding intercepts are proportional*).

Substitute m, n, and p for their equals AC, CE, and EF.

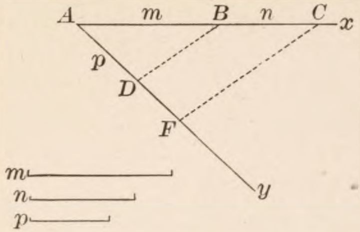
Then $$\frac{AH}{m}=\frac{HK}{n}=\frac{KB}{p}.$$

Q.E.F.

Ex. 286. Divide a line 12 inches long into three parts proportional to the numbers 3, 5, 7.

Proposition XXXVII. Problem.

386. *To find the fourth proportional to three given straight lines.*



Let the three given lines be m, n, and p.

To find the fourth proportional to m, n, and p.

Draw Ax and Ay containing any convenient angle.

On Ax take AB equal to m, BC to n.

On Ay take AD equal to p.

Draw BD.

From C draw $CF \parallel$ to BD, meeting Ay at F.

DF is the fourth proportional required.

Proof. $AB : BC = AD : DF$, § 342

(*a line drawn through two sides of a* $\triangle \parallel$ *to the third side divides those sides proportionally*).

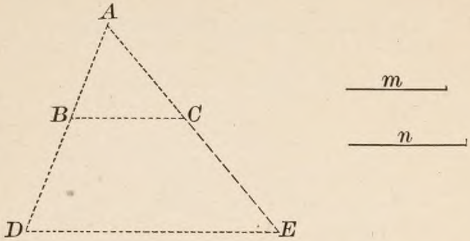
Substitute m, n, and p for their equals AB, BC, and AD.

Then $m : n = p : DF$. Q.E.F.

Ex. 287. The square of the altitude of an equilateral triangle is equal to three fourths of the square of one side of the triangle.

Proposition XXXVIII. Problem.

387. *To find the third proportional to two given straight lines.*



Let m and n be the two given straight lines.

To find the third proportional to m and n.

Construct any convenient angle A,

and take AB equal to m, AC equal to n.

Produce AB to D, making BD equal to AC.

Draw BC.

Through D draw $DE \parallel$ to BC, meeting AC produced at E.

CE is the third proportional required.

Proof. $AB : BD = AC : CE$, § 342

(*a line drawn through two sides of a* $\triangle$ *parallel to the third side divides those sides proportionally*).

Substitute, in the above proportion, AC for its equal BD.

Then $AB : AC = AC : CE$,

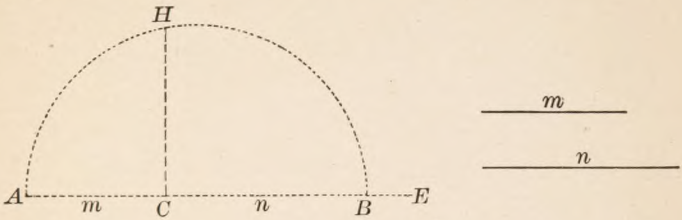
that is, $m : n = n : CE$. Q.E.F.

Ex. 288. Construct x, if (1) $x = \dfrac{ab}{c}$, (2) $x = \dfrac{a^2}{c}$.

Special cases: (1) $a = 2,\ b = 3,\ c = 4$; (2) $a = 3,\ b = 7,\ c = 11$; (3) $a = 2,\ c = 3$; (4) $a = 3,\ c = 5$; (5) $a = 2c$.

PROPOSITION XXXIX. PROBLEM.

388. *To find the mean proportional between two given straight lines.*



Let the two given lines be m and n.

To find the mean proportional between m and n.

On the straight line AE

take AC equal to m, and CB equal to n.

On AB as a diameter describe a semicircumference.

At C erect the $\perp$ CH meeting the circumference at H.

CH is the mean proportional between m and n.

Proof. $AC : CH = CH : CB$, § 370

(*the $\perp$ let fall from a point in a circumference to the diameter of a circle is the mean proportional between the segments of the diameter*).

Substitute for AC and CB their equals m and n.

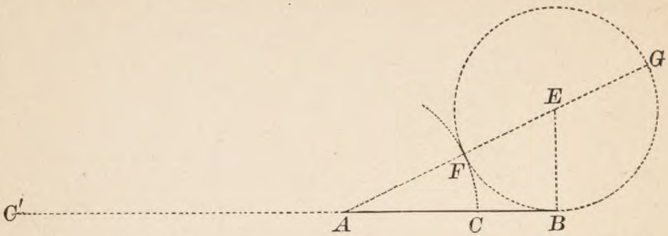
Then $m : CH = CH : n$. Q.E.F.

389. DEF. A straight line is divided **in extreme and mean ratio,** when one of the segments is the mean proportional between the whole line and the other segment.

Ex. 289. Construct x, if $x = \sqrt{ab}$.
Special cases: (1) $a = 2$, $b = 3$; (2) $a = 1$, $b = 5$; (3) $a = 3$, $b = 7$.

PROPOSITION XL. PROBLEM.

390. *To divide a given line in extreme and mean ratio.*



Let AB be the given line.

To divide AB in extreme and mean ratio.

At B erect a $\perp$ BE equal to half of AB.

From E as a centre, with a radius equal to EB, describe a $\odot$.

Draw AE, meeting the circumference in F and G.

On AB take AC equal to AF.

On BA produced take AC' equal to AG.

Then AB is divided internally at C and externally at C' in extreme and mean ratio.

Proof. $AG : AB = AB : AF$. § 381

$$\begin{aligned}\overline{AB}^2 &= AF \times AG\\ &= AC(AF + FG)\\ &= AC(AC + AB)\\ &= \overline{AC}^2 + AB \times AC.\end{aligned}$$

$$\therefore \overline{AB}^2 - AB \times AC = \overline{AC}^2.$$

$$\therefore AB(AB - AC) = \overline{AC}^2.$$

$$\therefore AB \times CB = \overline{AC}^2.$$

$$\begin{aligned}\overline{AB}^2 &= AG \times AF\\ &= C'A(AG - FG)\\ &= C'A(C'A - AB)\\ &= \overline{C'A}^2 - AB \times C'A.\end{aligned}$$

$$\therefore \overline{AB}^2 + AB \times C'A = \overline{C'A}^2.$$

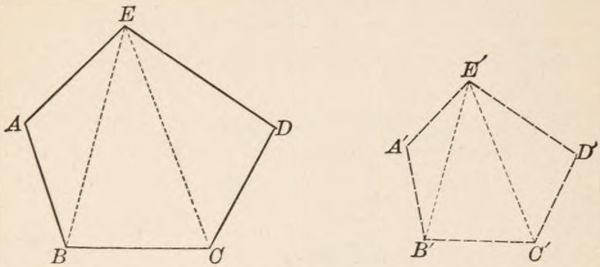
$$\therefore AB(AB + C'A) = \overline{C'A}^2.$$

$$\therefore AB \times C'B = \overline{C'A}^2.$$

Q. E. F.

PROPOSITION XLI. PROBLEM.

391. *Upon a given line homologous to a given side of a given polygon, to construct a polygon similar to the given polygon.*



Let A′E′ be the given line homologous to AE of the given polygon ABCDE.

To construct on $A'E'$ a polygon similar to the given polygon.

From E draw the diagonals EB and EC.

From E' draw $E'B'$, $E'C'$, and $E'D'$,

making $\angle s\ A'E'B'$, $B'E'C'$, and $C'E'D'$ equal, respectively, to $\angle s\ AEB$, BEC, and CED.

From A' draw $A'B'$, making $\angle E'A'B'$ equal to $\angle EAB$, and meeting $E'B'$ at B'.

From B' draw $B'C'$, making $\angle E'B'C'$ equal to $\angle EBC$, and meeting $E'C'$ at C'.

From C' draw $C'D'$, making $\angle E'C'D'$ equal to $\angle ECD$, and meeting $E'D'$ at D'.

Then $A'B'C'D'E'$ is the required polygon.

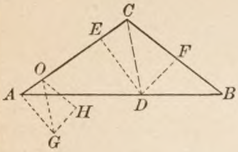
Proof. The $\triangle s\ ABE$, $A'B'E'$, etc., are similar. § 354

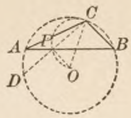
Therefore, the two polygons are similar. § 366

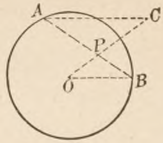
Q. E. F.

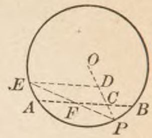
PROBLEMS OF CONSTRUCTION.

Ex. 290. To divide one side of a given triangle into segments proportional to the adjacent sides (§ 348).









Ex. 291. To find in one side of a given triangle a point whose distances from the other sides shall be to each other in the given ratio $m : n$.

Take $AG = m \perp$ to AC, $GH = n \perp$ to BC. Draw $CD \parallel$ to OG.

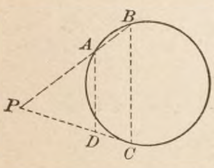
Ex. 292. Given an obtuse triangle; to draw a line from the vertex of the obtuse angle to the opposite side which shall be the mean proportional between the segments of that side.

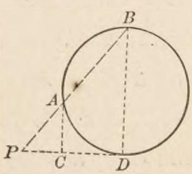
Ex. 293. Through a given point P within a given circle to draw a chord AB so that the ratio $AP : BP$ shall equal the given ratio $m : n$.

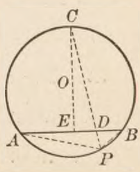
Draw OPC so that $OP : PC = n : m$. Draw CA equal to the fourth proportional to n, m, and the radius of the circle.

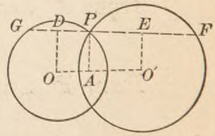
Ex. 294. To draw through a given point P in the arc subtended by a chord AB a chord which shall be bisected by AB.

On radius OP take CD equal to CP. Draw $DE \parallel$ to BA.









Ex. 295. To draw through a given external point P a secant PAB to a given circle so that the ratio $PA : AB$ shall equal the given ratio $m : n$.

$$PD : DC = m : n. \qquad PD : PA = PA : PC.$$

Ex. 296. To draw through a given external point P a secant PAB to a given circle so that $\overline{AB}^2 = PA \times PB$.

$$PC : CD = CD : PD. \qquad PA = CD.$$

Ex. 297. To find a point P in the arc subtended by a given chord AB so that the ratio $PA : PB$ shall equal the given ratio $m : n$.

Ex. 298. To draw through one of the points of intersection of two circles a secant so that the two chords that are formed shall be in the given ratio $m : n$.

Ex. 299. Having given the greater segment of a line divided in extreme and mean ratio, to construct the line.

Ex. 300. To construct a circle which shall pass through two given points and touch a given straight line.

Ex. 301. To construct a circle which shall pass through a given point and touch two given straight lines.

Ex. 302. To inscribe a square in a semicircle.

Ex. 303. To inscribe a square in a given triangle.

Let $DEFG$ be the required inscribed square. Draw $CM \parallel$ to AB, meeting AF produced in M. Draw CH and $MN \perp$ to AB, and produce AB to meet MN at N. The $\triangle ACM$, AGF are similar; also, the $\triangle AMN$, AFE are similar. By these triangles show that the figure $CMNH$ is a square. By constructing this square, the point F can be found.

Ex. 304. To inscribe in a given triangle a rectangle similar to a given rectangle.

Ex. 305. To inscribe in a circle a triangle similar to a given triangle.

Ex. 306. To inscribe in a given semicircle a rectangle similar to a given rectangle.

Ex. 307. To circumscribe about a circle a triangle similar to a given triangle.

Ex. 308. To construct the expression, $x = \frac{2\,abc}{de}$; that is, $\frac{2\,ab}{d} \times \frac{c}{e}$.

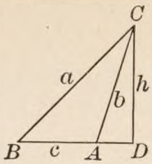
Ex. 309. To construct two straight lines, having given their sum and their ratio.

Ex. 310. To construct two straight lines, having given their difference and their ratio.

Ex. 311. Given two circles, with centres O and O', and a point A in their plane, to draw through the point A a straight line, meeting the circumferences at B and C, so that $AB : AC = m : n$.

PROBLEMS OF COMPUTATION.

Ex. 312. To compute the altitudes of a triangle in terms of its sides.



At least one of the angles A or B is acute. Suppose B is acute.

In the $\triangle CDB$, $\quad h^2 = a^2 - \overline{BD}^2.$ § 372

In the $\triangle ABC$, $\quad b^2 = a^2 + c^2 - 2c \times BD.$ § 375

Whence $\quad BD = \dfrac{a^2 + c^2 - b^2}{2c}.$

Hence,
$$h^2 = a^2 - \frac{(a^2 + c^2 - b^2)^2}{4c^2} = \frac{4a^2c^2 - (a^2 + c^2 - b^2)^2}{4c^2}$$
$$= \frac{(2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)}{4c^2}$$
$$= \frac{\{(a + c)^2 - b^2\}\{b^2 - (a - c)^2\}}{4c^2}$$
$$= \frac{(a + b + c)(a + c - b)(b + a - c)(b - a + c)}{4c^2}.$$

Let $\quad a + b + c = 2s.$

Then $\quad a + c - b = 2(s - b),$

$\quad b + a - c = 2(s - c),$

$\quad b - a + c = 2(s - a).$

Hence,
$$h^2 = \frac{2s \times 2(s - a) \times 2(s - b) \times 2(s - c)}{4c^2}.$$

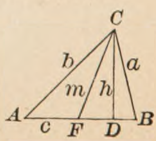
By simplifying, and extracting the square root,
$$h = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

Ex. 313. To compute the medians of a triangle in terms of its sides.

By § 377, $\quad a^2 + b^2 = 2m^2 + 2\left(\dfrac{c}{2}\right)^2.$

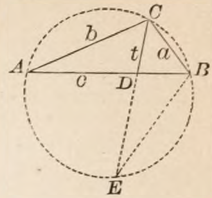
Whence $\quad 4m^2 = 2(a^2 + b^2) - c^2.$

$\quad \therefore m = \frac{1}{2}\sqrt{2(a^2 + b^2) - c^2}.$



Ex. 314. To compute the bisectors of a triangle in terms of the sides.

By § 383, $t^2 = ab - AD \times BD.$



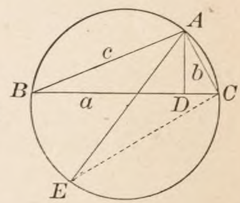
By § 348, $\dfrac{AD}{b} = \dfrac{BD}{a} = \dfrac{AD + BD}{a+b} = \dfrac{c}{a+b}.$

$$\therefore AD = \frac{bc}{a+b}, \text{ and } BD = \frac{ac}{a+b}.$$

Whence
$$t^2 = ab - \frac{abc^2}{(a+b)^2}$$
$$= ab\left[1 - \frac{c^2}{(a+b)^2}\right]$$
$$= \frac{ab\{(a+b)^2 - c^2\}}{(a+b)^2}$$
$$= \frac{ab(a+b+c)(a+b-c)}{(a+b)^2}$$
$$= \frac{ab \times 2s \times 2(s-c)}{(a+b)^2}.$$

Whence
$$t = \frac{2}{a+b}\sqrt{abs(s-c)}.$$

Ex. 315. To compute the radius of the circle circumscribed about a triangle in terms of the sides of the triangle.



By § 384, $AC \times AB = AE \times AD,$

or, $bc = 2R \times AD.$

But $AD = \dfrac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$ Ex. 312

$$\therefore R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

Ex. 316. If the sides of a triangle are 3, 4, and 5, is the angle opposite 5 right, acute, or obtuse?

Ex. 317. If the sides of a triangle are 7, 9, and 12, is the angle opposite 12 right, acute, or obtuse?

Ex. 318. If the sides of a triangle are 7, 9, and 11, is the angle opposite 11 right, acute, or obtuse?

Ex. 319. The legs of a right triangle are 8 inches and 12 inches; find the lengths of the projections of these legs upon the hypotenuse, and the distance of the vertex of the right angle from the hypotenuse.

Ex. 320. If the sides of a triangle are 6 inches, 9 inches, and 12 inches, find the lengths (1) of the altitudes; (2) of the medians; (3) of the bisectors; (4) of the radius of the circumscribed circle.

Ex. 321. A line is drawn parallel to a side AB of a triangle ABC, cutting AC in D, BC in E. If $AD : DC = 2 : 3$, and $AB = 20$ inches, find DE.

Ex. 322. The sides of a triangle are 9, 12, 15. Find the segments of the sides made by bisecting the angles.

Ex. 323. A tree casts a shadow 90 feet long, when a post 6 feet high casts a shadow 4 feet long. How high is the tree?

Ex. 324. The lower and upper bases of a trapezoid are a, b, respectively; and the altitude is h. Find the altitudes of the two triangles formed by producing the legs until they meet.

Ex. 325. The sides of a triangle are 6, 7, 8, respectively. In a similar triangle the side homologous to 8 is 40. Find the other two sides.

Ex. 326. The perimeters of two similar polygons are 200 feet and 300 feet. If a side of the first is 24 feet, find the homologous side of the second.

Ex. 327. How long a ladder is required to reach a window 24 feet high, if the lower end of the ladder is 10 feet from the side of the house?

Ex. 328. If the side of an equilateral triangle is a, find the altitude.

Ex. 329. If the altitude of an equilateral triangle is h, find the side.

Ex. 330. Find the length of the longest chord and of the shortest chord that can be drawn through a point 6 inches from the centre of a circle whose radius is 10 inches.

Ex. 331. The distance from the centre of a circle to a chord 10 feet long is 12 feet. Find the distance from the centre to a chord 24 feet long.

Ex. 332. The radius of a circle is 5 inches. Through a point 3 inches from the centre a diameter is drawn, and also a chord perpendicular to the diameter. Find the length of this chord, and the distance from one end of the chord to the ends of the diameter.

Ex. 333. The radius of a circle is 6 inches. Find the lengths of the tangents drawn from a point 10 inches from the centre, and also the length of the chord joining the points of contact.

Ex. 334. The sides of a triangle are 407 feet, 368 feet, and 351 feet. Find the three bisectors and the three altitudes.

Ex. 335. If a chord 8 inches long is 3 inches distant from the centre of the circle, find the radius, and the chords drawn from the end of the chord to the ends of the diameter which bisects the chord.

Ex. 336. From the end of a tangent 20 inches long a secant is drawn through the centre of the circle. If the external segment of this secant is 8 inches, find the radius of the circle.

Ex. 337. The radius of a circle is 13 inches. Through a point 5 inches from the centre any chord is drawn. What is the product of the two segments of the chord? What is the length of the shortest chord that can be drawn through the point?

Ex. 338. The radius of a circle is 9 inches and the length of a tangent 12 inches. Find the length of a line drawn from the extremity of the tangent to the centre of the circle.

Ex. 339. Two circles have radii of 8 inches and 3 inches, respectively, and the distance between their centres is 15 inches. Find the lengths of their common tangents.

Ex. 340. Find the segments of a line 10 inches long divided in extreme and mean ratio.

Ex. 341. The sides of a triangle are 4, 5, 5. Is the largest angle acute, right, or obtuse?

Ex. 342. Find the third proportional to two lines whose lengths are 28 feet and 42 feet.

Ex. 343. If the sides of a triangle are a, b, c, respectively, find the lengths of the three altitudes.

Ex. 344. The diameter of a circle is 30 feet and is divided into five equal parts. Find the lengths of the chords drawn through the points of division perpendicular to the diameter.

Ex. 345. The radius of a circle is 2 inches. From a point 4 inches from the centre a secant is drawn so that the internal segment is 1 inch. Find the length of the secant.

Ex. 346. The sides of a triangular pasture are 1551 yards, 2068 yards, 2585 yards. Find the median to the longest side.

Ex. 347. The diagonal of a rectangle is d, and the perimeter is p. Find the sides.

Ex. 348. The radius of a circle is r. Find the length of a chord whose distance from the centre is $\frac{1}{2}r$.

Book IV.

AREAS OF POLYGONS.

392. Def. The **unit of surface** is a square whose side is a *unit of length.*

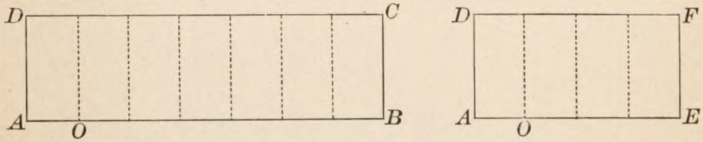
393. Def. The **area of a surface** is the *number of units of surface* it contains.

394. Def. Plane figures that *have equal areas but cannot be made to coincide* are called **equivalent.**

Note. In propositions relating to *areas*, the words "rectangle," "triangle," etc., are often used for "area of rectangle," "area of triangle," etc.

Proposition I. Theorem.

395. *The areas of two rectangles having equal altitudes are to each other as their bases.*



Let the rectangles AC and AF have the same altitude AD.

To prove that rect. AC : *rect.* AF = *base* AB : *base* AE.

Case 1. *When* AB *and* AE *are commensurable.*

Proof. Suppose AB and AE have a common measure, as AO, which is contained m times in AB and n times in AE.

Then $$AB : AE = m : n.$$

Apply AO as a unit of measure to AB and AE, and at the several points of division erect $\perp$s.

The rect. AC is divided into m rectangles,

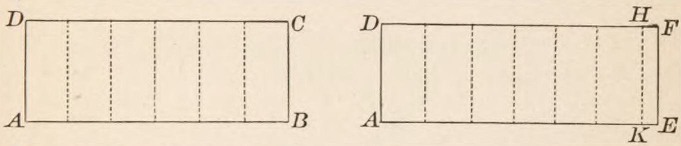
and the rect. AF is divided into n rectangles. § 107

These rectangles are all equal. § 186

Hence, rect. AC : rect. $AF = m : n$.

Therefore, rect. AC : rect. $AF = AB : AE$. Ax. 1

CASE 2. *When AB and AE are incommensurable.*



Proof. Divide AB into any number of equal parts, and apply one of them to AE as many times as AE will contain it.

Since AB and AE are incommensurable, a certain number of these parts will extend from A to some point K, leaving a remainder KE less than one of the equal parts of AB.

Draw $KH \parallel$ to EF.

Then AB and AK are commensurable by construction.

Therefore, $$\frac{\text{rect. } AH}{\text{rect. } AC} = \frac{AK}{AB}.$$ Case 1

If the number of equal parts into which AB is divided is indefinitely increased, the varying values of these ratios will continue equal, and approach for their respective limits the ratios

$$\frac{\text{rect. } AF}{\text{rect. } AC} \text{ and } \frac{AE}{AB}. \quad \text{(See § 287.)}$$

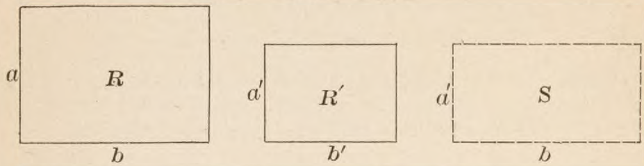
$$\therefore \frac{\text{rect. } AF}{\text{rect. } AC} = \frac{AE}{AB}.$$ § 284

Q. E. D.

396. COR. *The areas of two rectangles having equal bases are to each other as their altitudes.*

PROPOSITION II. THEOREM.

397. *The areas of two rectangles are to each other as the products of their bases by their altitudes.*



Let R and R′ be two rectangles, having for their bases b and b′, and for their altitudes a and a′, respectively.

To prove that $$\frac{R}{R'} = \frac{a \times b}{a' \times b'}.$$

Proof. Construct the rectangle S, with its base equal to that of R, and its altitude equal to that of R'.

Then $$\frac{R}{S} = \frac{a}{a'},$$ § 396

and $$\frac{S}{R'} = \frac{b}{b'}.$$ § 395

The products of the corresponding members of these equations give $$\frac{R}{R'} = \frac{a \times b}{a' \times b'}.$$

Q.E.D.

Ex. 349. Find the ratio of a rectangular lawn 72 yards by 49 yards to a grass turf 18 inches by 14 inches.

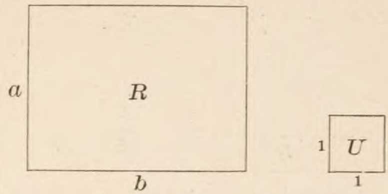
Ex. 350. Find the ratio of a rectangular courtyard $18\frac{1}{2}$ yards by $15\frac{1}{2}$ yards to a flagstone 31 inches by 18 inches.

Ex. 351. A square and a rectangle have the same perimeter, 100 yards. The length of the rectangle is 4 times its breadth. Compare their areas.

Ex. 352. On a certain map the linear scale is 1 inch to 5 miles. How many acres are represented on this map by a square the perimeter of which is 1 inch?

Proposition III. Theorem.

398. *The area of a rectangle is equal to the product of its base by its altitude.*



Let R be a rectangle, b its base, and a its altitude.

To prove that $\quad$ *the area of* $R = a \times b$.

Proof. Let U be the unit of surface.

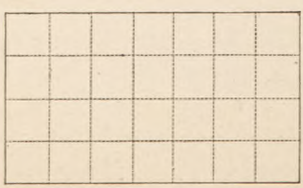
$$\frac{R}{U} = \frac{a \times b}{1 \times 1} = a \times b, \qquad \S\ 397$$

(*two rectangles are to each other as the products of their bases and altitudes*).

But $\quad \dfrac{R}{U} =$ the *number* of units of surface in R. $\qquad$ § 393

$$\therefore \text{ the area of } R = a \times b. \qquad \text{Q. E. D.}$$

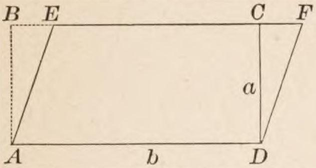
399. Scholium. When the base and altitude each contain the linear unit an integral number of times, this proposition is rendered evident by dividing the figure into squares, each

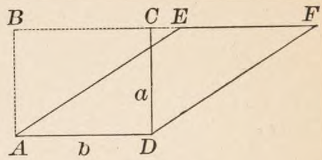


equal to the unit of surface. Thus, if the base contains seven linear units, and the altitude four, the figure may be divided into twenty-eight squares, each equal to the unit of surface.

PROPOSITION IV. THEOREM.

400. *The area of a parallelogram is equal to the product of its base by its altitude.*





Let AEFD be a parallelogram, b its base, and a its altitude.

To prove that the area of the $\square$ $AEFD = a \times b$.

Proof. From A draw $AB \parallel$ to DC to meet FE produced.

Then the figure $ABCD$ is a rectangle, with the same base and the same altitude as the $\square$ $AEFD$.

The rt. $\triangle$ ABE and DCF are equal. § 151

For $AB = CD$, and $AE = DF$. § 178

From $ABFD$ take the $\triangle$ DCF; the rect. $ABCD$ is left.

From $ABFD$ take the $\triangle$ ABE; the $\square$ $AEFD$ is left.

$\therefore$ rect. $ABCD \approx \square$ $AEFD$. Ax. 3

But the area of the rect. $ABCD = a \times b$. § 398

$\therefore$ the area of the $\square$ $AEFD = a \times b$. Ax. 1

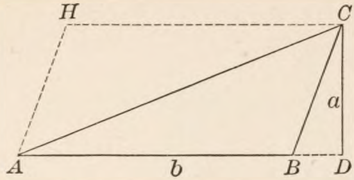
Q. E. D.

401. COR. 1. *Parallelograms having equal bases and equal altitudes are equivalent.*

402. COR. 2. *Parallelograms having equal bases are to each other as their altitudes; parallelograms having equal altitudes are to each other as their bases; any two parallelograms are to each other as the products of their bases by their altitudes.*

PROPOSITION V. THEOREM.

403. *The area of a triangle is equal to half the product of its base by its altitude.*



Let a be the altitude and b the base of the triangle ABC.

To prove that the area of the $\triangle ABC = \frac{1}{2} a \times b$.

Proof. Construct on AB and BC the parallelogram $ABCH$.

Then $\triangle ABC = \frac{1}{2} \square ABCH$. § 179

The area of the $\square ABCH = a \times b$. § 400

Therefore, the area of $\triangle ABC = \frac{1}{2} a \times b$. Ax. 7

Q. E. D.

404. COR. 1. *Triangles having equal bases and equal altitudes are equivalent.*

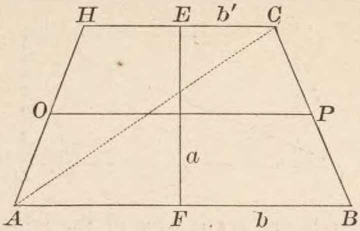
405. COR. 2. *Triangles having equal bases are to each other as their altitudes; triangles having equal altitudes are to each other as their bases; any two triangles are to each other as the products of their bases by their altitudes.*

406. COR. 3. *The product of the legs of a right triangle is equal to the product of the hypotenuse by the altitude from the vertex of the right angle.*

Ex. 353. The lines which join the middle point of either diagonal of a quadrilateral to the opposite vertices divide the quadrilateral into two equivalent parts.

PROPOSITION VI. THEOREM.

407. *The area of a trapezoid is equal to half the sum of its bases multiplied by the altitude.*



Let b and b′ be the bases and a the altitude of the trapezoid ABCH.

To prove that the area of $ABCH = \frac{1}{2}a(b + b')$.

Proof. Draw the diagonal AC.

Then the area of the $\triangle ABC = \frac{1}{2}a \times b$,

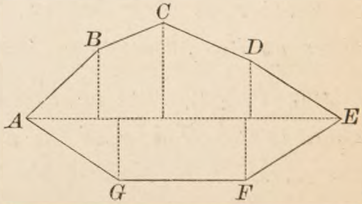
and the area of the $\triangle AHC = \frac{1}{2}a \times b'$. § 403

$\therefore$ the area of $ABCH = \frac{1}{2}a(b + b')$. Ax. 2

Q. E. D.

408. COR. *The area of a trapezoid is equal to the product of the median by the altitude.* § 190

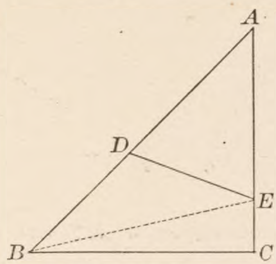
409. SCHOLIUM. The area of an irregular polygon may be found by dividing the polygon into triangles, and by finding the area of each of these triangles separately. Or, we may draw the longest diagonal, and let fall perpendiculars upon this diagonal from the other vertices of the polygon.



The sum of the areas of the right triangles, rectangles, and trapezoids thus formed is the area of the polygon.

PROPOSITION VII. THEOREM.

410. *The areas of two triangles which have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles.*



Let the triangles ABC and ADE have the common angle A.

To prove that $$\frac{\triangle ABC}{\triangle ADE} = \frac{AB \times AC}{AD \times AE}.$$

Proof. Draw BE.

Now $$\frac{\triangle ABC}{\triangle ABE} = \frac{AC}{AE},$$

and $$\frac{\triangle ABE}{\triangle ADE} = \frac{AB}{AD}. \qquad \text{§ 405}$$

The products of the first members and of the second members of these equalities give

$$\frac{\triangle ABC}{\triangle ADE} = \frac{AB \times AC}{AD \times AE}.$$

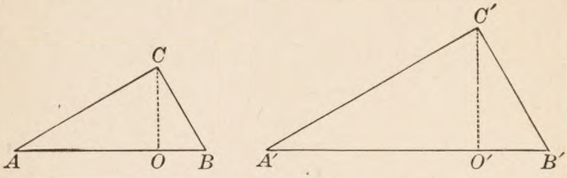
Q.E.D.

Ex. 354. The areas of two triangles which have an angle of the one supplementary to an angle of the other are to each other as the products of the sides including the supplementary angles.

COMPARISON OF POLYGONS.

Proposition VIII. Theorem.

411. *The areas of two similar triangles are to each other as the squares of any two homologous sides.*



Let the two similar triangles be ACB and A′C′B′.

To prove that $$\frac{\triangle ACB}{\triangle A'C'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Proof. Draw the altitudes CO and $C'O'$.

Then $$\frac{\triangle ACB}{\triangle A'C'B'} = \frac{AB \times CO}{A'B' \times C'O'} = \frac{AB}{A'B'} \times \frac{CO}{C'O'}, \qquad \S\ 405$$

(*two* $\triangle$ *are to each other as the products of their bases by their altitudes*).

But $$\frac{AB}{A'B'} = \frac{CO}{C'O'}, \qquad \S\ 361$$

(*the homologous altitudes of two similar* $\triangle$ *have the same ratio as any two homologous sides*).

Substitute, in the above equality, for $\frac{CO}{C'O'}$ its equal $\frac{AB}{A'B'}$;

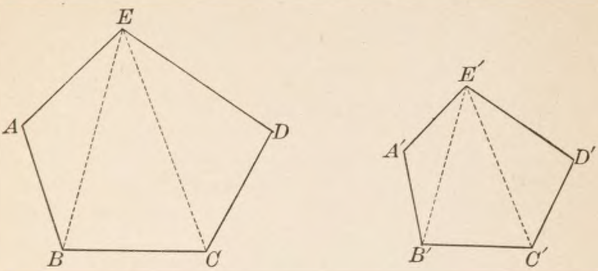
then $$\frac{\triangle ACB}{\triangle A'C'B'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Q.E.D.

Ex. 355. Prove this proposition by § 410.

PROPOSITION IX. THEOREM.

412. *The areas of two similar polygons are to each other as the squares of any two homologous sides.*



Let S and S′ denote the areas of the two similar polygons ABC etc. and A′B′C′ etc.

To prove that $S : S' = \overline{AB}^2 : \overline{A'B'}^2.$

Proof. By drawing all the diagonals from any homologous vertices E and E', the two similar polygons are divided into similar triangles. § 365

$$\therefore \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\triangle ABE}{\triangle A'B'E'} = \left(\frac{\overline{BE}^2}{\overline{B'E'}^2}\right) = \frac{\triangle BCE}{\triangle B'C'E'} = \text{etc.} \qquad \S\ 411$$

That is, $$\frac{\triangle ABE}{\triangle A'B'E'} = \frac{\triangle BCE}{\triangle B'C'E'} = \frac{\triangle CDE}{\triangle C'D'E'}.$$

$$\therefore \frac{\triangle ABE + \triangle BCE + \triangle CDE}{\triangle A'B'E' + \triangle B'C'E' + \triangle C'D'E'} = \frac{\triangle ABE}{\triangle A'B'E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad \S\ 335$$

$$\therefore S : S' = \overline{AB}^2 : \overline{A'B'}^2.$$ Q.E.D.

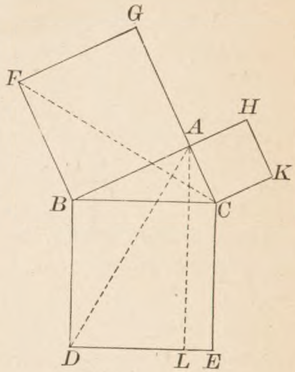
413. COR. 1. *The areas of two similar polygons are to each other as the squares of any two homologous lines.*

414. COR. 2. *The homologous sides of two similar polygons have the same ratio as the square roots of their areas.*

PROPOSITION X. THEOREM.

415. *The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the two legs.*

Let BE, CH, AF be squares on the three sides of the right triangle ABC.



To prove that $BE \approx CH + AF$.

Proof. Through A draw $AL \parallel$ to CE, and draw AD and CF.

Since $\angle s$ BAC, BAG, and CAH are rt. $\angle s$, CAG and BAH are straight lines. § 90

The $\triangle ABD = \triangle FBC$. § 143

For $BD = BC$,

$BA = BF$, § 168

and $\angle ABD = \angle FBC$, Ax. 2

(*each being the sum of a rt.* $\angle$ *and the* $\angle ABC$).

Now the rectangle BL is double the $\triangle ABD$,

(*having the same base BD, and the same altitude, the distance between the* $\parallel$s *AL and BD*),

and the square AF is double the $\triangle FBC$,

(*having the same base FB, and the same altitude AB*).

$\therefore$ the rectangle BL is equivalent to the square AF. Ax. 6

In like manner, by drawing AE and BK, it may be proved that the rectangle CL is equivalent to the square CH.

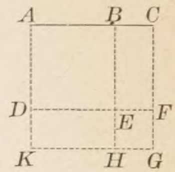
Hence, the square BE, the sum of the rectangles BL and CL, is equivalent to the sum of the squares CH and AF. Q.E.D.

416. COR. *The square on either leg of a right triangle is equivalent to the difference of the square on the hypotenuse and the square on the other leg.*

THEOREMS.

Ex. 356. The square constructed upon the sum of two straight lines is equivalent to the sum of the squares constructed upon these two lines, increased by twice the rectangle of these lines.

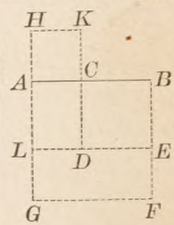
Let AB and BC be the two straight lines, and AC their sum. Construct the squares $ACGK$ and $ABED$ upon AC and AB, respectively. Prolong BE and DE until they meet KG and CG, respectively. Then we have the square $EFGH$, with sides each equal to BC. Hence, the square $ACGK$ is the sum of the squares $ABED$ and $EFGH$, and the rectangles $DEHK$ and $BCFE$, the dimensions of which are equal to AB and BC.



Ex. 357. The square constructed upon the difference of two straight lines is equivalent to the sum of the squares constructed upon these two lines, diminished by twice the rectangle of these lines.

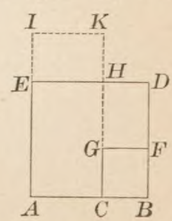
Let AB and AC be the two straight lines, and BC their difference. Construct the square $ABFG$ upon AB, the square $ACKH$ upon AC, and the square $BEDC$ upon BC (as shown in the figure). Prolong ED to meet AG in L.

The dimensions of the rectangles $LEFG$ and $HKDL$ are AB and AC, and the square $BCDE$ is evidently the difference between the whole figure and the sum of these rectangles; that is, the square constructed upon BC is equivalent to the sum of the squares constructed upon AB and AC, diminished by twice the rectangle of AB and AC.



Ex. 358. The difference between the squares constructed upon two straight lines is equivalent to the rectangle of the sum and difference of these lines.

Let $ABDE$ and $BCGF$ be the squares constructed upon the two straight lines AB and BC. The difference between these squares is the polygon $ACGFDE$, which is composed of the rectangles $ACHE$ and $GFDH$. Prolong AE and CH to I and K, respectively, making EI and HK each equal to BC, and draw IK. The rectangles $GFDH$ and $EHKI$ are equal. The difference between the squares $ABDE$ and $BCGF$ is then equivalent to the rectangle $ACKI$, which has for dimensions AI, equal to $AB + BC$, and EH, equal to $AB - BC$.



Ex. 359. The area of a rhombus is equal to half the product of its diagonals.

Ex. 360. Two isosceles triangles are equivalent if their legs are equal each to each, and the altitude of one is equal to half the base of the other.

Ex. 361. The area of a circumscribed polygon is equal to half the product of its perimeter by the radius of the inscribed circle.

Ex. 362. Two parallelograms are equal if two adjacent sides of the one are equal, respectively, to two adjacent sides of the other, and the included angles are supplementary.

Ex. 363. If ABC is a right triangle, C the vertex of the right angle, BD a line cutting AC in D, then $\overline{BD}^2 + \overline{AC}^2 = \overline{AB}^2 + \overline{DC}^2$.

Ex. 364. Upon the sides of a right triangle as homologous sides three similar polygons are constructed. Prove that the polygon upon the hypotenuse is equivalent to the sum of the polygons upon the legs.

Ex. 365. If the middle points of two adjacent sides of a parallelogram are joined, a triangle is formed which is equivalent to one eighth of the parallelogram.

Ex. 366. If any point within a parallelogram is joined to the four vertices, the sum of either pair of triangles having parallel bases is equivalent to half the parallelogram.

Ex. 367. Every straight line drawn through the intersection of the diagonals of a parallelogram divides the parallelogram into two equal parts.

Ex. 368. The line which joins the middle points of the bases of a trapezoid divides the trapezoid into two equivalent parts.

Ex. 369. Every straight line drawn through the middle point of the median of a trapezoid cutting both bases divides the trapezoid into two equivalent parts.

Ex. 370. If two straight lines are drawn from the middle point of either leg of a trapezoid to the opposite vertices, the triangle thus formed is equivalent to half the trapezoid.

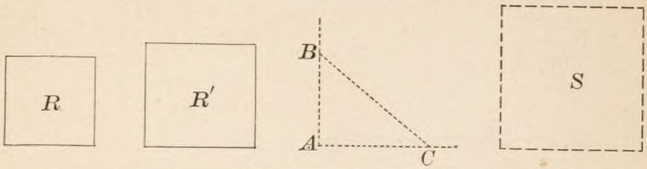
Ex. 371. The area of a trapezoid is equal to the product of one of the legs by the distance from this leg to the middle point of the other leg.

Ex. 372. The figure whose vertices are the middle points of the sides of any quadrilateral is equivalent to half the quadrilateral.

PROBLEMS OF CONSTRUCTION.

Proposition XI. Problem.

417. *To construct a square equivalent to the sum of two given squares.*



Let R and R′ be two given squares.

To construct a square equivalent to $R' + R$.

Construct the rt. $\angle A$.

Take AC equal to a side of R',

and AB equal to a side of R; and draw BC.

Construct the square S, having each of its sides equal to BC.

Then S is the square required.

Proof. $\overline{BC}^2 \Leftrightarrow \overline{AC}^2 + \overline{AB}^2$, § 415

(*the square on the hypotenuse of a rt.* $\triangle$ *is equivalent to the sum of the squares on the two legs*).

$\therefore S \Leftrightarrow R' + R$.

Q. E. F.

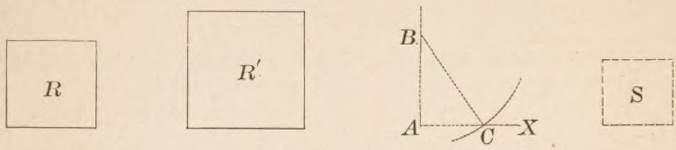
Ex. 373. If the perimeter of a rectangle is 72 feet, and the length is equal to twice the width, find the area.

Ex. 374. How many tiles 9 inches long and 4 inches wide will be required to pave a path 8 feet wide surrounding a rectangular court 120 feet long and 36 feet wide?

Ex. 375. The bases of a trapezoid are 16 feet and 10 feet; each leg is equal to 5 feet. Find the area of the trapezoid.

PROPOSITION XII. PROBLEM.

418. *To construct a square equivalent to the difference of two given squares.*



Let R be the smaller square and R′ the larger.

To construct a square equivalent to $R' - R$.

Construct the rt. $\angle A$.

Take AB equal to a side of R.

From B as a centre, with a radius equal to a side of R', describe an arc cutting the line AX at C.

Construct the square S, having each of its sides equal to AC.

Then S is the square required.

Proof. $\overline{AC}^2 \approx \overline{BC}^2 - \overline{AB}^2$, § 416

(*the square on either leg of a rt.* $\triangle$ *is equivalent to the difference of the square on the hypotenuse and the square on the other leg*).

$\therefore S \approx R' - R$. Q.E.F.

Ex. 376. Construct a square equivalent to the sum of two squares whose sides are 3 inches and 4 inches.

Ex. 377. Construct a square equivalent to the difference of two squares whose sides are $2\frac{1}{2}$ inches and 2 inches.

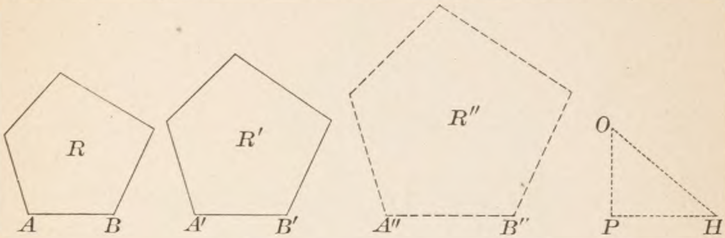
Ex. 378. Find the side of a square equivalent to the sum of two squares whose sides are 24 feet and 32 feet.

Ex. 379. Find the side of a square equivalent to the difference of two squares whose sides are 24 feet and 40 feet.

Ex. 380. A rhombus contains 100 square feet, and the length of one diagonal is 10 feet. Find the length of the other diagonal.

PROPOSITION XIII. PROBLEM.

419. *To construct a polygon similar to two given similar polygons and equivalent to their sum.*



Let R and R′ be two similar polygons, and AB and A′B′ two homologous sides.

To construct a similar polygon equivalent to $R + R'$.

Construct the rt. $\angle P$.

Take PH equal to $A'B'$, and PO equal to AB.

Draw OH, and take $A''B''$ equal to OH.

Upon $A''B''$, homologous to AB, construct R'' similar to R.

Then R'' is the polygon required.

Proof. $\overline{PO}^2 + \overline{PH}^2 = \overline{OH}^2$. § 415

Put for PO, PH, and OH their equals AB, $A'B'$, and $A''B''$.

Then $\overline{AB}^2 + \overline{A'B'}^2 = \overline{A''B''}^2$.

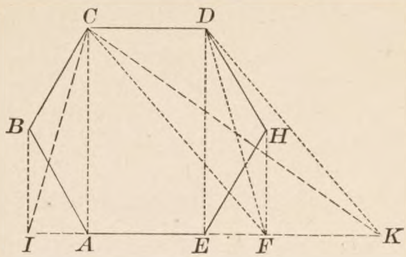
Now $\dfrac{R}{R''} = \dfrac{\overline{AB}^2}{\overline{A''B''}^2}$, and $\dfrac{R'}{R''} = \dfrac{\overline{A'B'}^2}{\overline{A''B''}^2}$. § 412

By addition, $\dfrac{R + R'}{R''} = \dfrac{\overline{AB}^2 + \overline{A'B'}^2}{\overline{A''B''}^2} = 1$. Ax. 2

$\therefore R'' \approx R + R'$. Q. E. F.

Proposition XIV. Problem.

420. *To construct a triangle equivalent to a given polygon.*



Let ABCDHE be the given polygon.

To construct a triangle equivalent to the given polygon.

Let D, H, and E be any three consecutive vertices of the polygon. Draw the diagonal DE.

From H draw $HF \parallel$ to DE.

Produce AE to meet HF at F, and draw DF.

Again, draw CF, and draw $DK \parallel$ to CF to meet AF produced at K, and draw CK.

In like manner continue to reduce the number of sides of the polygon until we obtain the $\triangle\ CIK$.

Then $\triangle\ CIK$ is the triangle required.

Proof. The polygon $ABCDF$ has one side less than the polygon $ABCDHE$, but the two polygons are equivalent.

For the part $ABCDE$ is common,

and the $\triangle\ DEF \approx \triangle\ DEH$, § 404

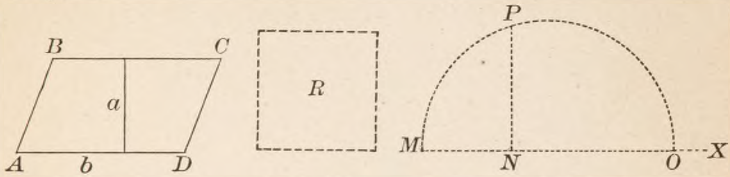
(*for the base DE is common, and their vertices F and H are in the line FH* $\parallel$ *to the base*).

In like manner it may be proved that

$ABCK \approx ABCDF$, and $CIK \approx ABCK$. Q. E. F.

PROPOSITION XV. PROBLEM.

421. *To construct a square equivalent to a given parallelogram.*



Let ABCD be the parallelogram, b its base, and a its altitude.

To construct a square equivalent to the $\square$ *ABCD.*

Upon a line MX take MN equal to a, NO equal to b.

Upon MO as a diameter, describe a semicircle.

At N erect $NP \perp$ to MO, meeting the circumference at P.

Then the square R, constructed upon a line equal to NP, is equivalent to the $\square$ $ABCD$.

Proof. $MN : NP = NP : NO,$ § 370

(*a* $\perp$ *let fall from any point of a circumference to the diameter is the mean proportional between the segments of the diameter*).

$$\therefore \overline{NP}^2 = MN \times NO = a \times b. \qquad \S\ 327$$

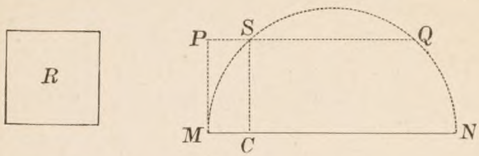
Therefore, $R \Leftrightarrow \square\ ABCD$. Q.E.F.

422. COR. 1. *A square may be constructed equivalent to a given triangle, by taking for its side the mean proportional between the base and half the altitude of the triangle.*

423. COR. 2. *A square may be constructed equivalent to a given polygon, by first reducing the polygon to an equivalent triangle, and then constructing a square equivalent to the triangle.*

PROPOSITION XVI. PROBLEM.

424. *To construct a parallelogram equivalent to a given square, and having the sum of its base and altitude equal to a given line.*



Let R be the given square, and let the sum of the base and altitude of the required parallelogram be equal to the given line MN.

To construct a $\square$ *equivalent to* R, *with the sum of its base and altitude equal to* MN.

Upon MN as a diameter, describe a semicircle.

At M erect MP, a $\perp$ to MN, equal to a side of the given square R.

Draw $PQ \parallel$ to MN, cutting the circumference at S.

Draw $SC \perp$ to MN.

Any $\square$ having CM for its altitude and CN for its base is equivalent to R.

Proof. $SC = PM$. §§ 104, 180

$$\therefore \overline{SC}^2 = \overline{PM}^2 = R.$$

But $MC : SC = SC : CN$, § 370

(*a* $\perp$ *let fall from any point of a circumference to the diameter is the mean proportional between the segments of the diameter*).

Then $\overline{SC}^2 \approx MC \times CN$. § 327

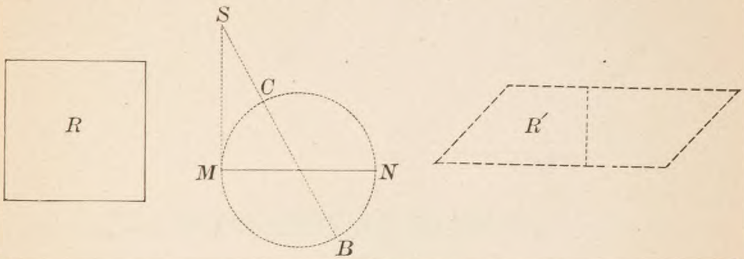
Q. E. F.

NOTE. This problem may be stated as follows:

To construct two straight lines the sum and product of which are known.

PROPOSITION XVII. PROBLEM.

425. *To construct a parallelogram equivalent to a given square, and having the difference of its base and altitude equal to a given line.*



Let R be the given square, and let the difference of the base and altitude of the required parallelogram be equal to the given line MN.

To construct a ▱ equivalent to R, with the difference of the base and altitude equal to MN.

Upon the given line MN as a diameter, describe a circle.

From M draw MS, tangent to the ⊙, and equal to a side of the given square R.

Through the centre of the ⊙ draw SB intersecting the circumference at C and B.

Then any ▱, as R', having SB for its base and SC for its altitude, is equivalent to R.

Proof. $$SB : SM = SM : SC, \qquad \text{§ 381}$$

(*if from a point without a ⊙ a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and the external segment*).

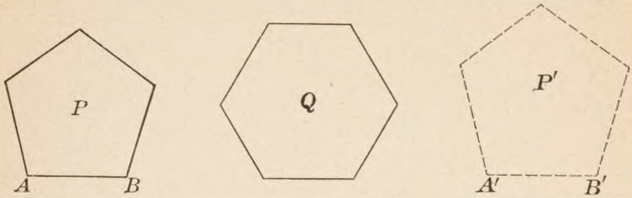
Then $$\overline{SM}^2 \approx SB \times SC, \qquad \text{§ 327}$$

and the difference between SB and SC is the diameter of the ⊙, that is, MN. Q.E.F.

NOTE. This problem may be stated: *To construct two straight lines the difference and product of which are known.*

PROPOSITION XVIII. PROBLEM.

426. *To construct a polygon similar to a given polygon P and equivalent to a given polygon Q.*



Let P and Q be the two given polygons, and AB a side of P.

To construct a polygon similar to P and equivalent to Q.

Find squares equivalent to P and Q, § 423

and let m and n respectively denote their sides.

Find $A'B'$, the fourth proportional to m, n, and AB. § 386

Upon $A'B'$, homologous to AB, construct P' similar to P.

Then $P' \Leftrightarrow Q$.

Proof. $m : n = AB : A'B'$. Const.

$\therefore m^2 : n^2 = \overline{AB}^2 : \overline{A'B'}^2$. § 338

But $P \Leftrightarrow m^2$, and $Q \Leftrightarrow n^2$. Const.

$\therefore P : Q = m^2 : n^2 = \overline{AB}^2 : \overline{A'B'}^2$.

But $P : P' = \overline{AB}^2 : \overline{A'B'}^2$. § 412

$\therefore P : Q = P : P'$. Ax. 1

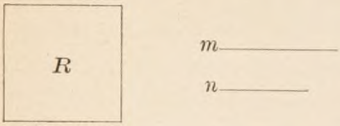
$\therefore P' \Leftrightarrow Q$. Q.E.F.

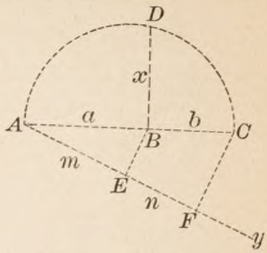
Ex. 381. To construct a square equivalent to the sum of any number of given squares.

Ex. 382. To construct a polygon similar to two given similar polygons and equivalent to their difference.

PROPOSITION XIX. PROBLEM.

427. *To construct a square which shall have a given ratio to a given square.*





Let R be the given square, and $\frac{n}{m}$ the given ratio.

To construct a square which shall be to R as n is to m.

Take AB equal to a side of R, and draw Ay, making any convenient angle with AB.

On Ay take AE equal to m, EF equal to n, and draw EB.

Draw $FC \parallel$ to EB meeting AB produced at C.

On AC as a diameter, describe a semicircle.

At B erect the $\perp$ BD, meeting the semicircumference at D.

Then BD is a side of the square required.

Proof. Denote AB by a, BC by b, and BD by x.

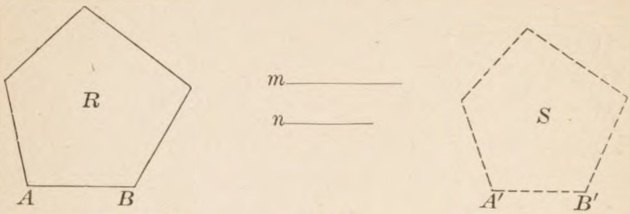
Now	$a : x = x : b.$	§ 370
Therefore,	$a^2 : x^2 = a : b.$	§ 337
But	$a : b = m : n.$	§ 342
Therefore,	$a^2 : x^2 = m : n.$	Ax. 1
By inversion,	$x^2 : a^2 = n : m.$	§ 331

Hence, the square on BD will have the same ratio to R as n has to m.

Q.E.F

Proposition XX. Problem.

428. *To construct a polygon similar to a given polygon and having a given ratio to it.*



Let R be the given polygon, and $\frac{n}{m}$ the given ratio.

To construct a polygon similar to R, which shall be to R as n is to m.

Construct a line $A'B'$, such that the square on $A'B'$ shall be to the square on AB as n is to m. § 427

Upon $A'B'$, as a side homologous to AB, construct the polygon S similar to R. § 391

Then S is the polygon required.

Proof. $S : R = \overline{A'B'}^2 : \overline{AB}^2$. § 412

But $\overline{A'B'}^2 : \overline{AB}^2 = n : m$. Const.

Therefore, $S : R = n : m$. Ax. 1

Q. E. F.

Ex. 383. To construct a triangle equivalent to a given triangle, and having one side equal to a given length l.

Ex. 384. To transform a triangle into an equivalent right triangle.

Ex. 385. To transform a given triangle into an equivalent right triangle, having one leg equal to a given length.

Ex. 386. To transform a given triangle into an equivalent right triangle, having the hypotenuse equal to a given length.

PROBLEMS OF CONSTRUCTION.

Ex. 387. To transform a triangle ABC into an equivalent triangle, having a side equal to a given length l, and an angle equal to angle BAC.

Upon AB (produced if necessary), take AD equal to l, draw $BE \parallel$ to CD, meeting AC (produced if necessary) at E. $\triangle BED \approx \triangle BEC$.

Ex. 388. To transform a given triangle into an equivalent isosceles triangle, having the base equal to a given length.

To construct a triangle equivalent to:

Ex. 389. The sum of two given triangles.

Ex. 390. The difference of two given triangles.

Ex. 391. To transform a given triangle into an equivalent equilateral triangle. (see Ex. 395)

To transform a parallelogram into an equivalent:

Ex. 392. Parallelogram having one side equal to a given length.

Ex. 393. Parallelogram having one angle equal to a given angle.

Ex. 394. Rectangle having a given altitude.

To transform a square into an equivalent:

Ex. 395. Equilateral triangle.

Ex. 396. Right triangle having one leg equal to a given length.

Ex. 397. Rectangle having one side equal to a given length.

To construct a square equivalent to:

Ex. 398. Five eighths of a given square.

Ex. 399. Three fifths of a given pentagon. or of a square similar to the given pentagon.

Ex. 400. To divide a given triangle into two equivalent parts by a line through a given point P in one of the sides.

Ex. 401. To find a point within a triangle, such that the lines joining this point to the vertices shall divide the triangle into three equivalent parts.

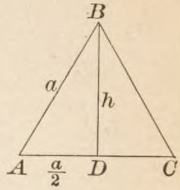
Ex. 402. To divide a given triangle into two equivalent parts by a line parallel to one of the sides.

Ex. 403. To divide a given triangle into two equivalent parts by a line perpendicular to one of the sides.

PROBLEMS OF COMPUTATION.

Ex. 404. To find the area of an equilateral triangle in terms of its side.

Denote the side by a, the altitude by h, and the area by S.



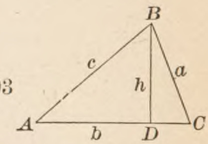
Then $$h^2 = a^2 - \frac{a^2}{4} = \frac{3\,a^2}{4} = \frac{a^2}{4} \times 3. \quad \S\ 372$$

$$\therefore h = \frac{a}{2}\sqrt{3}.$$

But $$S = \frac{a \times h}{2}. \quad \S\ 403$$

$$\therefore S = \frac{a}{2} \times \frac{a\sqrt{3}}{2} = \frac{a^2\sqrt{3}}{4}.$$

Ex. 405. To find the area of a triangle in terms of its sides.



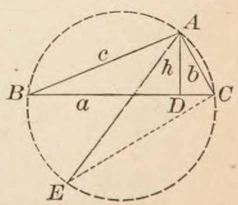
By Ex. 312, $h = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.$

Hence, $$S = \frac{b}{2} \times \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)} \quad \S 403$$
$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

Ex. 406. To find the area of a triangle in terms of the radius of the circumscribed circle.

If R denotes the radius of the circumscribed circle, and h the altitude of the triangle, we have, by § 384,

$$b \times c = 2\,R \times h.$$



Multiply by a, and we have,

$$a \times b \times c = 2\,R \times a \times h.$$

But $$a \times h = 2\,S. \quad \S\ 403$$

$$\therefore a \times b \times c = 4\,R \times S.$$

$$\therefore S = \frac{abc}{4\,R}.$$

Show that the radius of the circumscribed circle is equal to $\frac{abc}{4\,S}$.

Ex. 407. Find the area of a right triangle, if the length of the hypotenuse is 17 feet and the length of one leg is 8 feet.

Ex. 408. Find the ratio of the altitudes of two equivalent triangles, if the base of one is three times that of the other.

Ex. 409. The bases of a trapezoid are 8 feet and 10 feet, and the altitude is 6 feet. Find the base of the equivalent rectangle that has an equal altitude.

Ex. 410. Find the area of a rhombus, if the sum of its diagonals is 12 feet, and their ratio is 3 : 5.

Ex. 411. Find the area of an isosceles right triangle, if the hypotenuse is 20 feet.

Ex. 412. In a right triangle the hypotenuse is 13 feet, one leg is 5 feet. Find the area.

Ex. 413. Find the area of an isosceles triangle, if base $= b$, and leg $= c$.

Ex. 414. Find the area of an equilateral triangle, if one side $=$ 8 feet.

Ex. 415. Find the area of an equilateral triangle, if the altitude $= h$.

Ex. 416. A house is 40 feet long, 30 feet wide, 25 feet high to the roof, and 35 feet high to the ridge-pole. Find the number of square feet in its entire exterior surface.

Ex. 417. The sides of a right triangle are as 3 : 4 : 5. The altitude upon the hypotenuse is 12 feet. Find the area.

Ex. 418. Find the area of a right triangle, if one leg $= a$, and the altitude upon the hypotenuse $= h$.

Ex. 419. Find the area of a triangle, if the lengths of the sides are 104 feet, 111 feet, and 175 feet.

Ex. 420. The area of a trapezoid is 700 square feet. The bases are 30 feet and 40 feet, respectively. Find the altitude.

Ex. 421. $ABCD$ is a trapezium; $AB =$ 87 feet, $BC =$ 119 feet, $CD =$ 41 feet, $DA =$ 169 feet, $AC =$ 200 feet. Find the area.

Ex. 422. What is the area of a quadrilateral circumscribed about a circle whose radius is 25 feet, if the perimeter of the quadrilateral is 400 feet? What is the area of a hexagon that has a perimeter of 400 feet and is circumscribed about the same circle of 25 feet radius (Ex. 361)?

Ex. 423. The base of a triangle is 15 feet, and its altitude is 8 feet. Find the perimeter of an equivalent rhombus, if the altitude is 6 feet.

Ex. 424. Upon the diagonal of a rectangle 24 feet by 10 feet a triangle equivalent to the rectangle is constructed. What is its altitude?

Ex. 425. Find the side of a square equivalent to a trapezoid whose bases are 56 feet and 44 feet, and each leg is 10 feet.

Ex. 426. Through a point P in the side AB of a triangle ABC, a line is drawn parallel to BC so as to divide the triangle into two equivalent parts. Find the value of AP in terms of AB.

Ex. 427. What part of a parallelogram is the triangle cut off by a line from one vertex to the middle point of one of the opposite sides?

Ex. 428. In two similar polygons, two homologous sides are 15 feet and 25 feet. The area of the first polygon is 450 square feet. Find the area of the second polygon.

Ex. 429. The base of a triangle is 32 feet, its altitude 20 feet. What is the area of the triangle cut off by a line parallel to the base at a distance of 15 feet from the base?

Ex. 430. The sides of two equilateral triangles are 3 feet and 4 feet. Find the side of an equilateral triangle equivalent to their sum.

Ex. 431. If the side of one equilateral triangle is equal to the altitude of another, what is the ratio of their areas?

Ex. 432. The sides of a triangle are 10 feet, 17 feet, and 21 feet. Find the areas of the parts into which the triangle is divided by the bisector of the angle formed by the first two sides.

Ex. 433. In a trapezoid, one base is 10 feet, the altitude is 4 feet, the area is 32 square feet. Find the length of a line drawn between the legs parallel to the bases and distant 1 foot from the lower base.

Ex. 434. The diagonals of a rhombus are 90 yards and 120 yards, respectively. Find the area, the length of one side, and the perpendicular distance between two parallel sides.

Ex. 435. Find the number of square feet of carpet that are required to cover a triangular floor whose sides are, respectively, 26 feet, 35 feet, and 51 feet.

Ex. 436. If the altitude h of a triangle is increased by a length m, how much must be taken from the base a that the area may remain the same?

Ex. 437. Find the area of a right triangle, having given the segments p, q, into which the hypotenuse is divided by a perpendicular drawn to the hypotenuse from the vertex of the right angle.

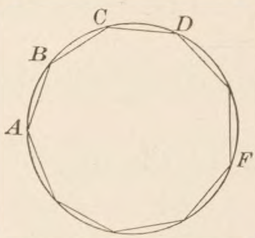
Book V.

REGULAR POLYGONS AND CIRCLES.

429. Def. A **regular polygon** is a polygon which is both equilateral and equiangular. The equilateral triangle and the square are examples.

Proposition I. Theorem.

430. *An equilateral polygon inscribed in a circle is a regular polygon.*



Let ABC etc. be an equilateral polygon inscribed in a circle.

To prove that the polygon ABC etc. is a regular polygon.

Proof. The arcs AB, BC, CD, etc., are equal. § 243

Hence, arcs ABC, BCD, etc., are equal. Ax. 2

Therefore, arcs CFA, DFB, etc., are equal. Ax. 3

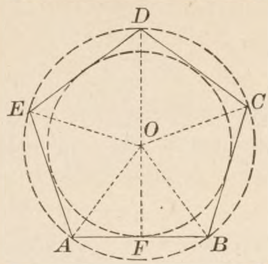
Therefore, $\angle s$ A, B, C, etc., are equal. § 289

Therefore, the polygon ABC etc. is a regular polygon, being equilateral and equiangular. § 429

Q. E. D.

PROPOSITION II. THEOREM.

431. *A circle may be circumscribed about, and a circle may be inscribed in, any regular polygon.*



Let ABCDE be a regular polygon.

1. *To prove that a circle may be circumscribed about $ABCDE$.*

Proof. Let O be the centre of the circle which may be passed through A, B, and C. § 258

Draw OA, OB, OC, and OD.

Then $\angle ABC = \angle BCD$, § 429

and $\angle OBC = \angle OCB$. § 145

By subtraction, $\angle OBA = \angle OCD$. Ax. 3

The $\triangle$ OBA and OCD are equal. § 143

For $\angle OBA = \angle OCD$,

$OB = OC$, § 217

and $AB = CD$. § 429

$\therefore OA = OD$. § 128

$\therefore$ the circle passing through A, B, C, passes through D.

In like manner it may be proved that the circle passing through B, C, and D also passes through E; and so on.

Therefore, the circle described from O as a centre, with a radius OA, will be circumscribed about the polygon. § 231

2. *To prove that a circle may be inscribed in $ABCDE$.*

Proof. Since the sides of the regular polygon are equal chords of the circumscribed circle, they are equally distant from the centre. § 249

Therefore, the circle described from O as a centre, with the distance from O to a side of the polygon as a radius, will be inscribed in the polygon (§ 232). Q. E. D.

432. Def. The radius of the circumscribed circle, OA, is called the **radius** of the polygon.

433. Def. The radius of the inscribed circle, OF, is called the **apothem** of the polygon.

434. Def. The common centre, O, of the circumscribed and inscribed circles is called the **centre** of the polygon.

435. Def. The angle between radii drawn to the extremities of any side is called the **angle at the centre** of the polygon.

By joining the centre to the vertices of a regular polygon, the polygon can be decomposed into as many equal isosceles triangles as it has sides.

436. Cor. 1. *The angle at the centre of a regular polygon is equal to four right angles divided by the number of sides of the polygon. Hence, the angles at the centre of any regular polygon are all equal.*

437. Cor. 2. *The radius drawn to any vertex of a regular polygon bisects the angle at the vertex.*

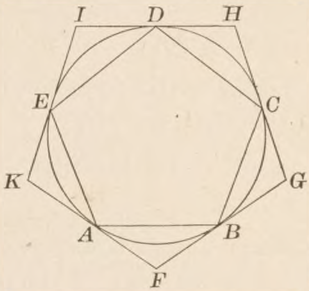
438. Cor. 3. *The angle at the centre of a regular polygon and an interior angle of the polygon are supplementary.*

For $\angle s$ FOB and FBO are complementary. § 135

$\therefore$ their doubles AOB and FBC are supplementary. Ax. 6

PROPOSITION III. THEOREM.

439. *If the circumference of a circle is divided into any number of equal arcs, the chords joining the successive points of division form a regular inscribed polygon; and the tangents drawn at the points of division form a regular circumscribed polygon.*



Suppose the circumference divided into equal arcs AB, BC, etc. Let AB, BC, etc., be the chords, FBG, GCH, etc., the tangents.

1. *To prove that $ABCDE$ is a regular polygon.*

Proof. The sides AB, BC, CD, etc., are equal. § 241

Therefore, the polygon is regular. § 430

2. *To prove that $FGHIK$ is a regular polygon.*

Proof. The ⩓ AFB, BGC, CHD, etc., are all equal isosceles triangles. §§ 295, 139

∴ ∡ F, G, H, etc., are equal, and FB, BG, GC, etc., are equal.

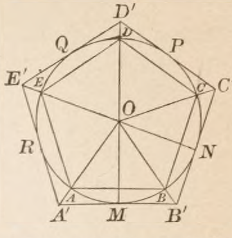
$\therefore FG = GH = HI$, etc. Ax. 6

$\therefore FGHIK$ is a regular polygon. § 429

Q. E. D.

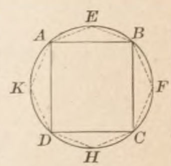
440. COR. 1. *Tangents to a circle at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides as the inscribed polygon.*

441. Cor. 2. *Tangents to a circle at the middle points of the arcs subtended by the sides of a regular inscribed polygon form a circumscribed regular polygon, whose sides are parallel to the sides of the inscribed polygon and whose vertices lie on the radii (prolonged) of the inscribed polygon.*

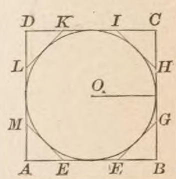


For two corresponding sides, AB and $A'B'$, are perpendicular to OM (§§ 248, 254), and are parallel (§ 104); and the tangents MB' and NB', intersecting at a point equidistant from OM and ON (§ 261), intersect upon the bisector of the $\angle MON$ (§ 162); that is, upon the radius OB. § 437

442. Cor. 3. *If the vertices of a regular inscribed polygon are joined to the middle points of the arcs subtended by the sides of the polygon, the joining lines form a regular inscribed polygon of double the number of sides.*



443. Cor. 4. *Tangents at the middle points of the arcs between adjacent points of contact of the sides of a regular circumscribed polygon form a regular circumscribed polygon of double the number of sides.*

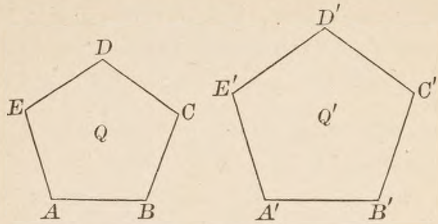


444. Cor. 5. *The perimeter of an inscribed polygon is less than the perimeter of an inscribed polygon of double the number of sides; and the perimeter of a circumscribed polygon is greater than the perimeter of a circumscribed polygon of double the number of sides.*

For two sides of a triangle are together greater than the third side. § 138

PROPOSITION IV. THEOREM.

445. *Two regular polygons of the same number of sides are similar.*



Let Q and Q′ be two regular polygons, each having n sides.

To prove that Q and Q' are similar.

Proof. The sum of the interior $\angle$s of each polygon is equal to

$$(n-2)2 \text{ rt. } \angle s, \qquad \S\ 205$$

(*the sum of the interior* $\angle$s *of a polygon is equal to* 2 *rt.* $\angle$s *taken as many times less two as the polygon has sides*).

$$\text{Each angle of either polygon} = \frac{(n-2)2 \text{ rt. } \angle s}{n}, \qquad \S\ 206$$

(*for the* $\angle$s *of a regular polygon are all equal, and hence each* $\angle$ *is equal to the sum of the* $\angle$s *divided by their number*).

Hence, the two polygons Q and Q' are mutually equiangular.

Since $AB = BC$, etc., and $A'B' = B'C'$, etc., § 429

$$AB : A'B' = BC : B'C', \text{ etc.}$$

Hence, the two polygons have their homologous sides proportional.

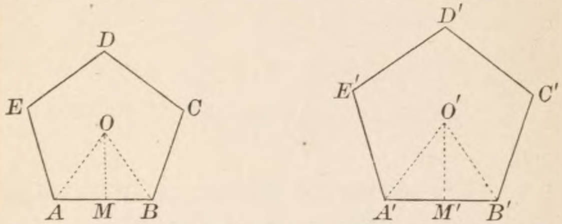
Therefore the two polygons are similar. § 351

Q. E. D.

446. COR. *The areas of two regular polygons of the same number of sides are to each other as the squares of any two homologous sides.* § 412

PROPOSITION V. THEOREM.

447. *The perimeters of two regular polygons of the same number of sides are to each other as the radii of their circumscribed circles, and also as the radii of their inscribed circles.*



Let P and P′ denote the perimeters, O and O′ the centres, of the two regular polygons.

From O, O' draw OA, $O'A'$, OB, $O'B'$, and the ⊥s OM, $O'M'$.

To prove that $P : P' = OA : O'A' = OM : O'M'$.

Proof. Since the polygons are similar, § 445

$$P : P' = AB : A'B'. \quad \text{§ 364}$$

The △s OAB and $O'A'B'$ are isosceles. § 431

Now $\angle O = \angle O'$, § 436

and $OA : OB = O'A' : O'B'$.

∴ the △s OAB and $O'A'B'$ are similar. § 357

$$\therefore AB : A'B' = OA : O'A'. \quad \text{§ 351}$$

Also, $AB : A'B' = OM : O'M'$. § 361

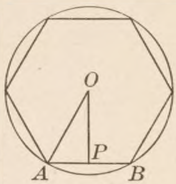
$$\therefore P : P' = OA : O'A' = OM : O'M'. \quad \text{Ax. 1}$$

Q. E. D.

448. COR. *The areas of two regular polygons of the same number of sides are to each other as the squares of the radii of the circumscribed circles, and of the inscribed circles.* § 413

Proposition VI. Theorem.

449. *If the number of sides of a regular inscribed polygon is indefinitely increased, the apothem of the polygon approaches the radius of the circle as its limit.*



Let AB be a side and OP the apothem of a regular polygon of n sides inscribed in the circle whose radius is OA.

To prove that OP approaches OA as a limit, when n increases indefinitely.

Proof. $OP < OA$, § 97

and $OA - OP < AP$. § 138

$\therefore OA - OP < AB$, which is twice AP. § 245

Now, if n is taken sufficiently great, AB, and consequently $OA - OP$, can be made less than any assigned value, however small, but cannot be made zero.

Since $OA - OP$ can be made less than any assigned value by increasing n, but cannot be made zero, OA is the limit of OP by the test for a limit. § 275

Q. E. D.

450. Cor. *If the number of sides of a regular inscribed polygon is indefinitely increased, the square of the apothem approaches the square of the radius of the circle as a limit.*

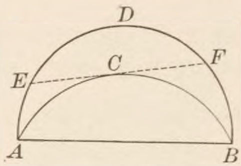
For $\overline{OA}^2 - \overline{OP}^2 = \overline{AP}^2$. § 372

But by taking n sufficiently great, AB and consequently AP, the half of AB, can be made less than any assigned value.

Therefore, $\overline{AP}^2$, the product of AP by AP, can be made less than any assigned value; for the product of two finite factors approaches zero as a limit, if *either* factor approaches zero as a limit (§ 276); and for a still stronger reason, the product approaches zero as a limit, if *each* of the factors approaches zero as a limit.

Proposition VII. Theorem.

451. *An arc of a circle is less than any line which envelops it and has the same extremities.*



Let ACB be an arc of a circle, and AB its chord.

To prove that the arc ACB is less than any other line which envelops this arc and terminates at A and B.

Proof. Of all the lines that can be drawn, each to include the area ACB between itself and the chord AB, there must be at least one shortest line; for all the lines are not equal.

Now the enveloping line ADB cannot be the shortest; for drawing ECF tangent to the arc ACB at C, the line $AECFB < AEDFB$, since $ECF < EDF$. § 49

In like manner it can be shown that no other enveloping line can be the shortest. Therefore, ACB is the shortest.

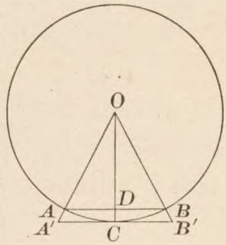
Q. E. D.

452. Cor. 1. *The circumference of a circle is less than the perimeter of any polygon circumscribed about it.*

453. Cor. 2. *Any convex curve is less than the perimeter of a polygon circumscribed about it.*

PROPOSITION VIII. THEOREM.

454. *The circumference of a circle is the limit which the perimeters of regular inscribed polygons and of similar circumscribed polygons approach, if the number of sides of the polygons is indefinitely increased; and the area of a circle is the limit which the areas of these polygons approach.*



Let P and P′ denote the lengths of the perimeters, AB and A′B′ two homologous sides, R and R′ the radii, of the polygons, and C the circumference of the circle.

1. *To prove that C is the limit of P and of P', if the number of sides of the polygons is indefinitely increased.*

Proof. Since the polygons are similar by hypothesis,

$$P' : P = R' : R. \qquad \text{§ 447}$$

Therefore, $P' - P : P = R' - R : R.$ § 333

Whence, $R(P' - P) = P(R' - R).$ § 327

Therefore, $$P' - P = \frac{P}{R}(R' - R).$$

Now P is always less than C. § 273

$$\therefore P' - P < \frac{C}{R}(R' - R).$$

But $R' - R$, which is less than $\frac{1}{2} A'C$ (§ 138), can be made less than any assigned quantity by increasing the number of sides of the polygons; and therefore $\frac{C}{R}(R' - R)$ can be made less than any assigned quantity. § 276

Hence, $P' - P$ can be made less than any assigned quantity.

Since P' is always greater than C (§ 452), and P is always less than C (§ 273), the difference between C and either P' or P is less than the difference $P' - P$, and consequently can be made less than any assigned quantity, but cannot be made zero.

Therefore, C is the common limit of P' and P. § 275

Let K denote the area of the circle, S the area of the inscribed polygon, and S′ the area of the circumscribed polygon.

2. *To prove that K is the limit of S and S'.*

Proof. $S' : S = R'^2 : R^2$. § 448

By division, $S' - S : S = R'^2 - R^2 : R^2$. § 333

Whence $$S' - S = \frac{S}{R^2}(R'^2 - R^2).$$

Now K is always greater than S. Ax. 8

Therefore, $$S' - S < \frac{K}{R^2}(R'^2 - R^2).$$

But $R'^2 - R^2$, which is equal to $(R' + R)(R' - R)$, can be made less than any assigned quantity; and therefore $\frac{K}{R^2}(R'^2 - R^2)$ can be made less than any assigned quantity. § 276

Hence, $S' - S$ can be made less than any assigned quantity.

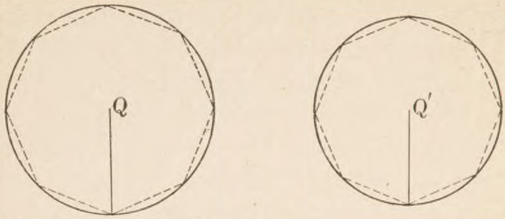
Since $S' > K$ always, and $S < K$ always (Ax. 8), the difference between K and either S' or S is less than the difference $S' - S$, and consequently can be made less than any assigned quantity, but cannot be made zero.

Therefore, K is the common limit of S' and S. § 275

Q. E. D.

PROPOSITION IX. THEOREM.

455. *Two circumferences have the same ratio as their radii.*



Let C and C′ be the circumferences, R and R′ the radii, of the two circles Q and Q′.

To prove that $C : C' = R : R'$.

Proof. Inscribe in the ⓢ two similar regular polygons, and denote their perimeters by P and P'.

Then $P : P' = R : R'$. § 447

Conceive the number of sides of these regular polygons to be indefinitely increased, the polygons continuing similar.

Then P and P' will have C and C' as limits. § 454

But $P : P'$ will always be equal to $R : R'$. § 447

$\therefore C : C' = R : R'$. § 285

Q. E. D.

456. COR. *The ratio of the circumference of a circle to its diameter is constant.*

For $C : C' = R : R'$. § 455

$\therefore C : C' = 2R : 2R'$. § 340

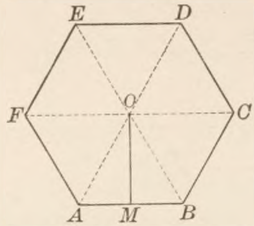
By alternation, $C : 2R = C' : 2R'$. § 330

457. DEF. The constant ratio of the circumference of a circle to its diameter is represented by the Greek letter π.

458. COR. $\pi = \dfrac{C}{2R}$. $\therefore C = 2\pi R$.

PROPOSITION X. THEOREM.

459. *The area of a regular polygon is equal to half the product of its apothem by its perimeter.*



Let P represent the perimeter, R the apothem, and S the area of the regular polygon ABC etc.

To prove that $S = \frac{1}{2} R \times P$.

Proof. Draw the radii OA, OB, OC, etc.

The polygon is divided into as many $\triangle$ as it has sides.

The apothem is the common altitude of these $\triangle$,

and the area of each $\triangle = \frac{1}{2} R$ multiplied by the base. § 403

Hence, the area of all the $\triangle$ is equal to $\frac{1}{2} R$ multiplied by the sum of all the bases.

But the sum of the areas of all the $\triangle$ is equal to the area of the polygon. Ax. 9

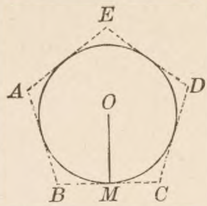
And the sum of all the bases of the $\triangle$ is equal to the perimeter of the polygon. Ax. 9

$$\therefore S = \frac{1}{2} R \times P.$$ Q.E.D.

460. DEF. In different circles **similar arcs, similar sectors,** and **similar segments** are such as correspond to *equal angles at the centre.*

Proposition XI. Theorem.

461. *The area of a circle is equal to half the product of its radius by its circumference.*



Let R represent the radius, C the circumference, and S the area, of the circle whose centre is O.

To prove that $S = \frac{1}{2} R \times C.$

Proof. Circumscribe any regular polygon about the circle, and denote its perimeter by P, and its area by S'.

Then $S' = \frac{1}{2} R \times P.$ § 459

Conceive the number of sides of the polygon to be indefinitely increased.

Then P approaches C as its limit, § 454

and S' approaches S as its limit. § 454

But $S' = \frac{1}{2} R \times P$, always. § 459

$\therefore S = \frac{1}{2} R \times C.$ § 284

Q. E. D.

462. Cor. 1. *The area of a sector is equal to half the product of its radius by its arc.*

For the sector and its arc are like parts of the circle and its circumference, respectively.

463. Cor. 2. *The area of a circle is equal to π times the square of its radius.*

For the area of the $\odot = \frac{1}{2} R \times C = \frac{1}{2} R \times 2\pi R = \pi R^2.$

464. Cor. 3. *The areas of two circles are to each other as the squares of their radii.*

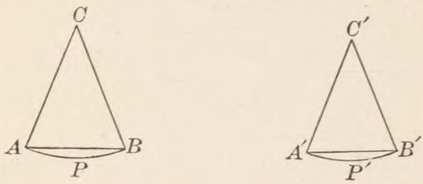
For, if S and S' denote the areas, and R and R' the radii,

$$S : S' = \pi R^2 : \pi R'^2 = R^2 : R'^2.$$

465. Cor. 4. *Similar arcs are to each other as their radii; similar sectors are to each other as the squares of their radii.*

Proposition XII. Theorem.

466. *The areas of two similar segments are to each other as the squares of their radii.*



Let AC and A′C′ be the radii of the two similar sectors ACB and A′C′B′, and let ABP and A′B′P′ be the corresponding segments.

To prove that $ABP : A'B'P' = \overline{AC}^2 : \overline{A'C'}^2.$

Proof. Sector ACB : sector $A'C'B' = \overline{AC}^2 : \overline{A'C'}^2$. § 465

The $\triangle$ ACB and $A'C'B'$ are similar. § 357

$\therefore \triangle ACB : \triangle A'C'B' = \overline{AC}^2 : \overline{A'C'}^2.$ § 411

$\therefore$ sector ACB : sector $A'C'B' = \triangle ACB : \triangle A'C'B'$. Ax. 1

$\therefore$ sector $ACB : \triangle ACB =$ sector $A'C'B' : \triangle A'C'B'$. § 330

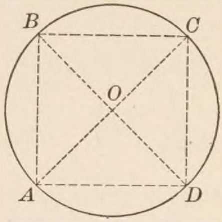
$$\therefore \frac{\text{sector } ACB - \triangle ACB}{\text{sector } A'C'B' - \triangle A'C'B'} = \frac{\triangle ACB}{\triangle A'C'B'} = \frac{\overline{AC}^2}{\overline{A'C'}^2}. \quad \S\ 333$$

That is, $ABP : A'B'P' = \overline{AC}^2 : \overline{A'C'}^2.$ Q. E. D.

PROBLEMS OF CONSTRUCTION.

Proposition XIII. Problem.

467. *To inscribe a square in a given circle.*



Let O be the centre of the given circle.

To inscribe a square in the given circle.

Draw two diameters AC and $BD \perp$ to each other.

Draw AB, BC, CD, and DA.

Then $ABCD$ is the square required.

Proof. The $\triangle ABC$, BCD, etc., are rt. $\triangle$, § 290

(*each being inscribed in a semicircle*),

and the sides AB, BC, etc., are equal, § 241

(*in the same ⊙ equal arcs are subtended by equal chords*).

Hence the quadrilateral $ABCD$ is a square. § 168

Q. E. F.

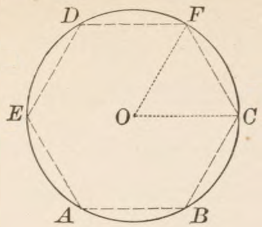
468. Cor. *By bisecting the arcs AB, BC, etc., a regular polygon of eight sides may be inscribed in the circle; and, by continuing the process, regular polygons of sixteen, thirty-two sixty-four, etc., sides may be inscribed.*

Ex. 438. The area of a circumscribed square is equal to twice the area of the inscribed square.

Ex. 439. The area of a circular ring is equal to that of a circle whose diameter is a chord of the outer circle tangent to the inner circle.

PROPOSITION XIV. PROBLEM.

469. *To inscribe a regular hexagon in a given circle.*



Let O be the centre of the given circle.

To inscribe a regular hexagon in the given circle.

From O draw any radius, as OC.

From C as a centre, with a radius equal to OC, describe an arc intersecting the circumference at F.

Draw OF and CF.

Then CF is a side of the regular hexagon required.

Proof. The $\triangle\ OFC$ is equiangular, § 146

(since it is equilateral by construction).

Hence, the $\angle\ FOC$ is $\frac{1}{3}$ of 2 rt. $\angle$s, or $\frac{1}{6}$ of 4 rt. $\angle$s. § 136

$\therefore$ the arc FC is $\frac{1}{6}$ of the circumference,

and the chord FC is a side of a regular inscribed hexagon.

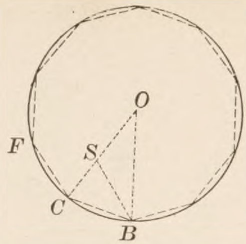
Hence, to inscribe a regular hexagon apply the radius six times as a chord. Q. E. F.

470. COR. 1. *By joining the alternate vertices A, C, D, an equilateral triangle is inscribed in the circle.*

471. COR. 2. *By bisecting the arcs AB, BC, etc., a regular polygon of twelve sides may be inscribed in the circle; and, by continuing the process, regular polygons of twenty-four, forty-eight, etc., sides may be inscribed.*

PROPOSITION XV. PROBLEM.

472. *To inscribe a regular decagon in a given circle.*



Let O be the centre of the given circle.

To inscribe a regular decagon in the given circle.

Draw any radius OC,

and divide it in extreme and mean ratio, so that OC shall be to OS as OS is to SC. § 389

From C as a centre, with a radius equal to OS,

describe an arc intersecting the circumference at B.

Draw BC.

Then BC is a side of the regular decagon required.

Proof. Draw BS and BO.

Now $OC : OS = OS : SC$, Const.

and $BC = OS$. Const.

$\therefore OC : BC = BC : SC$.

Moreover, $\angle OCB = \angle SCB$. Iden.

Hence, the $\triangle$ OCB and BCS are similar. § 357

But the $\triangle\ OCB$ is isosceles. § 217

$\therefore \triangle\ BCS$, which is similar to the $\triangle\ OCB$, is isosceles,

and $CB = BS = SO$. § 120

$\therefore$ the $\triangle\, SOB$ is isosceles, and the $\angle\, O = \angle\, SBO$. § 145

But the ext. $\angle\, CSB = \angle\, O + \angle\, SBO = 2 \angle\, O$. § 137

Hence, $\angle\, SCB = 2 \angle\, O$,

and $\angle\, OBC = 2 \angle\, O$.

$\therefore$ the sum of the $\angle$s of the $\triangle\, OCB = 5 \angle\, O = 2$ rt. $\angle$s,

and $\angle\, O = \frac{1}{5}$ of 2 rt. $\angle$s, or $\frac{1}{10}$ of 4 rt. $\angle$s.

Therefore, the arc BC is $\frac{1}{10}$ of the circumference,

and the chord BC is a side of a regular inscribed decagon.

Therefore, to inscribe a regular decagon, divide the radius internally in extreme and mean ratio, and apply the greater segment ten times as a chord. Q. E. F.

473. Cor. 1. *By joining the alternate vertices of a regular inscribed decagon, a regular pentagon is inscribed.*

474. Cor. 2. *By bisecting the arcs BC, CF, etc., a regular polygon of twenty sides may be inscribed in the circle; and, by continuing the process, regular polygons of forty, eighty, etc., sides may be inscribed.*

If R denotes the radius of a regular inscribed polygon, r the apothem, a one side, A an interior angle, and C the angle at the centre, show that S = the area

Ex. 440. In a regular inscribed triangle $a = R\sqrt{3}$, $r = \frac{1}{2}R$, $A = 60°$, $C = 120°$. $S = \frac{3\sqrt{3}\,R^2}{4}$

Ex. 441. In an inscribed square $a = R\sqrt{2}$, $r = \frac{1}{2}R\sqrt{2}$, $A = 90°$, $C = 90°$. $S = 2R^2$

Ex. 442. In a regular inscribed hexagon $a = R$, $r = \frac{1}{2}R\sqrt{3}$, $A = 120°$, $C = 60°$. $S = \frac{3\sqrt{3}\,R^2}{2}$

Ex. 443. In a regular inscribed decagon

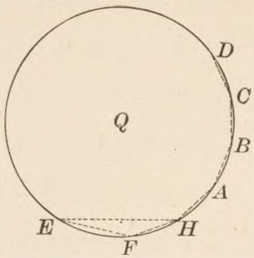
$$a = \frac{R(\sqrt{5}-1)}{2},\ r = \tfrac{1}{4}R\sqrt{10+2\sqrt{5}},\ A = 144°,\ C = 36°.$$

In a regular inscribed pentagon

$a = \frac{R}{2}\sqrt{10-2\sqrt{5}}$

PROPOSITION XVI. PROBLEM.

475. *To inscribe in a given circle a regular pentedecagon, or polygon of fifteen sides.*



Let Q be the given circle.

To inscribe in Q a regular pentedecagon.

Draw EH equal to the radius of the circle,

and EF equal to a side of the regular inscribed decagon. § 472

Draw FH.

Then FH is a side of the regular pentedecagon required.

Proof. The arc EH is $\frac{1}{6}$ of the circumference, § 469

and the arc EF is $\frac{1}{10}$ of the circumference. Const.

Hence, the arc FH is $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$, of the circumference.

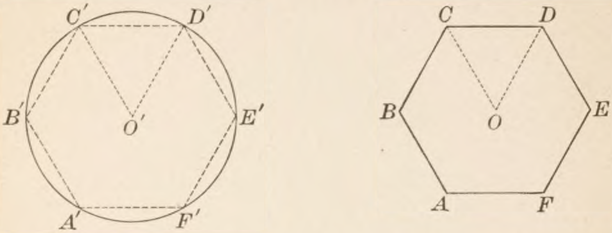
And the chord FH is a side of a regular inscribed pentedecagon.

By applying FH fifteen times as a chord, we have the polygon required. Q.E.F.

476. COR. *By bisecting the arcs FH, HA, etc., a regular polygon of thirty sides may be inscribed; and, by continuing the process, regular polygons of sixty, one hundred twenty, etc., sides may be inscribed.*

PROPOSITION XVII. PROBLEM.

477. *To inscribe in a given circle a regular polygon similar to a given regular polygon.*



Let ABC etc. be the given regular polygon, and O′ the centre of the given circle.

To inscribe in the circle a regular polygon similar to ABC etc.

From O, the centre of the given polygon,

draw OD and OC.

From O', the centre of the given circle,

draw $O'C'$ and $O'D'$,

making the $\angle O'$ equal to the $\angle O$.

Draw $C'D'$.

Then $C'D'$ is a side of the regular polygon required.

Proof. Each polygon has as many sides as the $\angle O$, or $\angle O'$, is contained times in 4 rt. $\angle$s.

Therefore, the polygon $C'D'E'$ etc. is similar to the polygon CDE etc., § 445

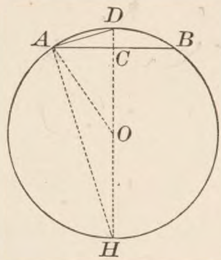
(*two regular polygons of the same number of sides are similar*).

Q. E. F.

Ex. 444. The area of an inscribed regular octagon is equal to that of the rectangle whose sides are equal to the sides of the inscribed and the circumscribed squares.

PROPOSITION XVIII. PROBLEM.

478. *Given the side and the radius of a regular inscribed polygon, to find the side of the regular inscribed polygon of double the number of sides.*



Let AB be a side of the regular inscribed polygon.

To find AD, a side of the regular inscribed polygon of double the number of sides.

Denote the radius by R, and AB by a.

From D draw DH through the centre O, and draw OA, AH.

DH is $\perp$ to AB at its middle point C. § 161

In the rt. $\triangle\, OCA$, $\overline{OC}^2 = R^2 - \frac{1}{4}a^2$. § 372

Therefore, $OC = \sqrt{R^2 - \frac{1}{4}a^2}$,

and $DC = R - \sqrt{R^2 - \frac{1}{4}a^2}$.

The $\angle\, DAH$ is a rt. $\angle$. § 290

In the rt. $\triangle\, DAH$, $\overline{AD}^2 = DH \times DC$. § 367

But $DH = 2\,R$, and $DC = R - \sqrt{R^2 - \frac{1}{4}a^2}$.

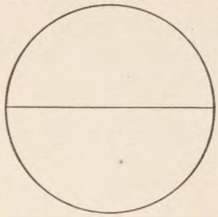
$$\therefore AD = \sqrt{2\,R\,(R - \sqrt{R^2 - \frac{1}{4}a^2})}$$

$$= \sqrt{R\,(2\,R - \sqrt{4\,R^2 - a^2})}. \quad \text{Q.E.F}$$

479. COR. *If* $R = 1$, $AD = \sqrt{2 - \sqrt{4 - a^2}}$.

PROPOSITION XIX. PROBLEM.

480. *To find the numerical value of the ratio of the circumference of a circle to its diameter.*



Let C be the circumference, when the radius is unity.

To find the numerical value of π.

By § 458, $2\pi R = C$. $\quad \therefore \pi = \frac{1}{2} C$ when $R = 1$.

Let S_6 be the length of a side of a regular polygon of 6 sides, S_{12} of 12 sides, and so on.

If $R = 1$, by § 469, $S_6 = 1$ and by § 479 we have

Form of Computation.	Length of Side.	Length of Perimeter.
$S_{12} = \sqrt{2 - \sqrt{4 - 1^2}}$	0.51763809	6.21165708
$S_{24} = \sqrt{2 - \sqrt{4 - (0.51763809)^2}}$	0.26105238	6.26525722
$S_{48} = \sqrt{2 - \sqrt{4 - (0.26105238)^2}}$	0.13080626	6.27870041
$S_{96} = \sqrt{2 - \sqrt{4 - (0.13080626)^2}}$	0.06543817	6.28206396
$S_{192} = \sqrt{2 - \sqrt{4 - (0.06543817)^2}}$	0.03272346	6.28290510
$S_{384} = \sqrt{2 - \sqrt{4 - (0.03272346)^2}}$	0.01636228	6.28311544
$S_{768} = \sqrt{2 - \sqrt{4 - (0.01636228)^2}}$	0.00818121	6.28316941

$\therefore C = 6.28317$ approximately; that is, $\pi = 3.14159$ nearly.

Q.E.F.

481. SCHOLIUM. π is incommensurable. We generally take

$$\pi = 3.1416, \text{ and } \frac{1}{\pi} = 0.31831.$$

π = 3,1415926535.

3, 1 4 1 5 9 2 6 5 3 5

Que j'aime à faire apprendre un nombre utile aux sages

MAXIMA AND MINIMA.

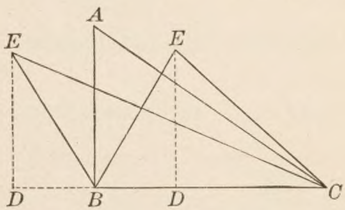
482. Def. Among geometrical magnitudes which satisfy given conditions, the *greatest* is called the **maximum**; and the *smallest* is called the **minimum**.

Thus, the diameter of a circle is the maximum among all chords; and the perpendicular is the minimum among all lines drawn to a given line from a given external point.

483. Def. **Isoperimetric** polygons are polygons which have equal perimeters.

Proposition XX. Theorem.

484. *Of all triangles having two given sides, that in which these sides include a right angle is the maximum.*



Let the triangles ABC and EBC have the sides AB and BC equal to EB and BC, respectively; and let the angle ABC be a right angle.

To prove that $\triangle ABC > \triangle EBC$.

Proof. From E draw the altitude ED.

The $\triangle$ ABC and EBC, having the same base, BC, are to each other as their altitudes AB and ED. § 405

Now $EB > ED$. § 97

But $EB = AB$. Hyp.

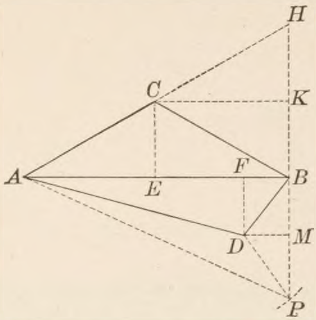
$\therefore AB > ED$.

$\therefore \triangle ABC > \triangle EBC$. § 405

Q. E. D.

PROPOSITION XXI. THEOREM.

485. *Of all isoperimetric triangles having the same base the isosceles triangle is the maximum.*



Let the $\triangle$ ACB and ADB have equal perimeters, and let AC and CB be equal, and AD and DB be unequal.

To prove that $\triangle ACB > \triangle ADB$.

Proof. Produce AC to H, making $CH = AC$; and draw HB.

Produce HB, take DP equal to DB, and draw AP.

Draw CE and $DF \perp$ to AB, and CK and $DM \parallel$ to AB.

The $\angle ABH$ is a right $\angle$, for it may be inscribed in the semicircle whose centre is C and radius CA. § 290

ADP is not a straight line, for then the $\angle$s DBA and DAB would be equal, being complements of the equal $\angle$s DBM and DPM, respectively; and DA and DB would be equal (§ 147), which is contrary to the hypothesis. Hence,

$AP < AD + DP, \therefore < AD + DB, \therefore < AC + CB, \therefore < AH.$

$\therefore BH > BP.$ § 102

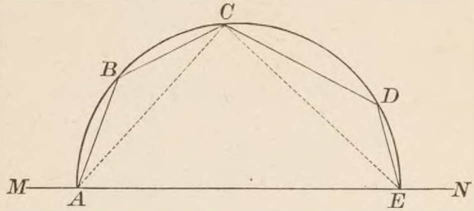
$\therefore CE (= \frac{1}{2} BH) > DF (= \frac{1}{2} BP).$ Ax. 7

Therefore, $\triangle ACB > \triangle ADB$. § 405

Q. E. D.

PROPOSITION XXII. THEOREM.

486. *Of all polygons with sides all given but one, the maximum can be inscribed in a semicircle which has the undetermined side for its diameter.*



Let ABCDE be the maximum of polygons with sides AB, BC, CD, DE, and the extremities A and E on the straight line MN.

To prove that $ABCDE$ can be inscribed in a semicircle.

Proof. From *any* vertex, as C, draw CA and CE.

The $\triangle ACE$ must be the maximum of all $\triangle$ having the sides CA and CE, and the third side on MN; otherwise, by increasing or diminishing the $\angle ACE$, keeping the lengths of the sides CA and CE unchanged, but sliding the extremities A and E along the line MN, we could increase the $\triangle ACE$, while the rest of the polygon would remain unchanged; and therefore increase the polygon. But this is contrary to the hypothesis that the polygon is the maximum polygon.

Hence, the $\triangle ACE$ is the maximum of $\triangle$ that have the sides CA and CE.

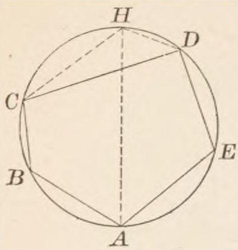
Therefore, the $\angle ACE$ is a right angle. § 484

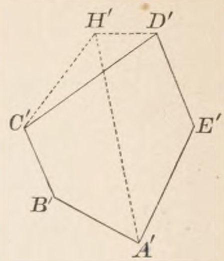
Therefore, C lies on the semicircumference. § 290

Hence, *every* vertex lies on the circumference; that is, the maximum polygon can be inscribed in a semicircle having the undetermined side for a diameter. Q. E. D.

PROPOSITION XXIII. THEOREM.

487. *Of all polygons with given sides, that which can be inscribed in a circle is the maximum.*





Let ABCDE be a polygon inscribed in a circle, and A′B′C′D′E′ be a polygon, equilateral with respect to ABCDE, which cannot be inscribed in a circle.

To prove that $ABCDE > A'B'C'D'E'$.

Proof. Draw the diameter AH, and draw CH and DH.

Upon $C'D'$ construct the $\triangle C'H'D' = \triangle CHD$, and draw $A'H'$.

Since, by hypothesis, a $\odot$ cannot pass through *all* the vertices of $A'B'C'H'D'E'$, *one* or *both* of the parts $ABCH$, $AEDH$ must be greater than the corresponding part of $A'B'C'H'D'E'$. § 486

If either of these parts is *not greater than* its corresponding part, it is equal to it, § 486

(*for* $ABCH$ *and* $AEDH$ *are the maxima of polygons that have sides equal to* AB, BC, CH, *and* AE, ED, DH, *respectively, and the remaining side undetermined*).

$$\therefore ABCHDE > A'B'C'H'D'E'. \qquad \text{Ax. 4}$$

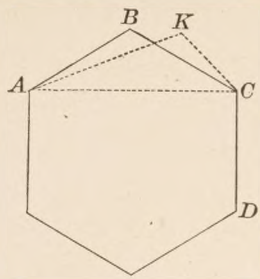
Take away from the two figures the equal $\triangle$ CHD and $C'H'D'$.

Then $$ABCDE > A'B'C'D'E'. \qquad \text{Ax. 5}$$

Q. E. D.

PROPOSITION XXIV. THEOREM.

488. *Of isoperimetric polygons of the same number of sides, the maximum is equilateral.*



Let ABCD etc. be the maximum of isoperimetric polygons of any given number of sides.

To prove that AB, BC, CD, *etc., are equal.*

Proof. Draw AC.

The $\triangle ABC$ must be the maximum of all the $\triangle$s which are formed upon AC with a perimeter equal to that of $\triangle ABC$.

Otherwise a greater $\triangle AKC$ could be substituted for $\triangle ABC$, without changing the perimeter of the polygon.

But this is inconsistent with the hypothesis that the polygon $ABCD$ etc. is the maximum polygon.

$\therefore$ the $\triangle ABC$ is isosceles. § 485

$$\therefore AB = BC.$$

In like manner it may be proved that $BC = CD$, etc. Q. E. D.

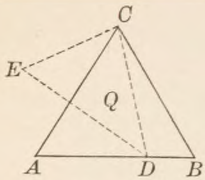
489. COR. *The maximum of isoperimetric polygons of the same number of sides is a regular polygon.*

For the maximum polygon is equilateral (§ 488), and can be inscribed in a circle (§ 487), and is, therefore, regular. § 430

Q. E. D.

PROPOSITION XXV. THEOREM.

490. *Of isoperimetric regular polygons, that which has the greatest number of sides is the maximum.*



Let Q be a regular polygon of three sides, and Q′ a regular polygon of four sides, and let the two polygons have equal perimeters.

To prove that *Q′ is greater than Q.*

Proof. Draw CD from C to any point in AB.

Invert the $\triangle\, CDA$ and place it in the position DCE, letting D fall at C, C at D, and A at E.

The polygon $DBCE$ is an irregular polygon of four sides, which by construction has the same perimeter as Q', and the same area as Q.

Then the irregular polygon $DBCE$ of four sides is less than the isoperimetric regular polygon Q' of four sides. § 489

In like manner it may be shown that Q' is less than an isoperimetric regular polygon of five sides, and so on. Q.E.D.

ΒΙΒΛΙΟΘΗΚΗ Μ. ΟΔΑΤΑΤΤΗΣ

Ex. 445. Of all equivalent parallelograms that have equal bases, the rectangle has the minimum perimeter.

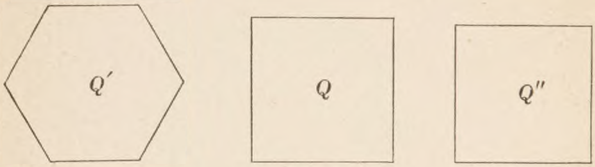
Ex. 446. Of all equivalent rectangles, the square has the minimum perimeter.

Ex. 447. Of all triangles that have the same base and the same altitude, the isosceles has the minimum perimeter.

Ex. 448. Of all triangles that can be inscribed in a given circle, the equilateral is the maximum and has the maximum perimeter.

PROPOSITION XXVI. THEOREM.

491. *Of regular polygons having a given area, that which has the greatest number of sides has the least perimeter.*



Let Q and Q′ be regular polygons having the same area, and let Q′ have the greater number of sides.

To prove the perimeter of $Q >$ the perimeter of Q'.

Proof. Let Q'' be a regular polygon having the same perimeter as Q', and the same number of sides as Q.

Then $Q' > Q''$, § 490

(*of isoperimetric regular polygons, that which has the greatest number of sides is the maximum*).

But $Q \approx Q'$. Hyp.

$\therefore Q > Q''$.

$\therefore$ the perimeter of $Q >$ the perimeter of Q''.

But the perimeter of $Q' =$ the perimeter of Q''. Hyp.

$\therefore$ the perimeter of $Q >$ the perimeter of Q'. Q. E. D.

Ex. 449. To inscribe in a semicircle the maximum rectangle.

Ex. 450. Of all polygons of a given number of sides which may be inscribed in a given circle, that which is regular has the maximum area and the maximum perimeter.

Ex. 451. Of all polygons of a given number of sides which may be circumscribed about a given circle, that which is regular has the minimum area and the minimum perimeter.

THEOREMS.

Ex. 452. Every equilateral polygon circumscribed about a circle is regular if it has an *odd* number of sides.

Ex. 453. Every equiangular polygon inscribed in a circle is regular if it has an *odd* number of sides.

Ex. 454. Every equiangular polygon circumscribed about a circle is regular.

Ex. 455. The side of a circumscribed equilateral triangle is equal to twice the side of the similar inscribed triangle.

Ex. 456. The apothem of an inscribed regular hexagon is equal to half the side of the inscribed equilateral triangle.

Ex. 457. The area of an inscribed regular hexagon is three fourths of the area of the circumscribed regular hexagon.

Ex. 458. The area of an inscribed regular hexagon is the mean proportional between the areas of the inscribed and the circumscribed equilateral triangles.

Ex. 459. The square of the side of an inscribed equilateral triangle is equal to three times the square of a side of the inscribed regular hexagon.

Ex. 460. The area of an inscribed equilateral triangle is equal to half the area of the inscribed regular hexagon.

Ex. 461. The square of the side of an inscribed equilateral triangle is equal to the sum of the squares of the sides of the inscribed square and of the inscribed regular hexagon.

Ex. 462. The square of the side of an inscribed regular pentagon is equal to the sum of the squares of the radius of the circle and the side of the inscribed regular decagon.

If R denotes the radius of a circle, and a one side of an inscribed regular polygon, show that:

Ex. 463. In a regular pentagon, $a = \frac{1}{4} R \sqrt{10 - 2\sqrt{5}}$.

Ex. 464. In a regular octagon, $a = R\sqrt{2 - \sqrt{2}}$.

Ex. 465. In a regular dodecagon, $a = R\sqrt{2 - \sqrt{3}}$.

Ex. 466. If two diagonals of a regular pentagon intersect, the longer segment of each is equal to a side of the pentagon.

Ex. 467. The apothem of an inscribed regular pentagon is equal to half the sum of the radius of the circle and the side of the inscribed regular decagon.

Ex. 468. The side of an inscribed regular pentagon is equal to the hypotenuse of the right triangle which has for legs the radius of the circle and the side of the inscribed regular decagon.

Ex. 469. The radius of an inscribed regular polygon is the mean proportional between its apothem and the radius of the similar circumscribed regular polygon.

Ex. 470. If squares are constructed outwardly upon the six sides of a regular hexagon, the exterior vertices of these squares are the vertices of a regular dodecagon.

Ex. 471. If the alternate vertices of a regular hexagon are joined by straight lines, show that another regular hexagon is thereby formed. Find the ratio of the areas of these two hexagons.

Ex. 472. If on the legs of a right triangle as diameters semicircles are described external to the triangle, and from the whole figure a semicircle on the hypotenuse is subtracted, the remaining figure is equivalent to the given right triangle.

Ex. 473. The star-shaped polygon, formed by producing the sides of a regular hexagon, is equivalent to twice the given hexagon.

Ex. 474. The sum of the perpendiculars drawn to the sides of a regular polygon from any point within the polygon is equal to the apothem multiplied by the number of sides.

Ex. 475. If two chords of a circle are perpendicular to each other, the sum of the four circles described on the four segments as diameters is equivalent to the given circle. To prove that $AO^2+OB^2+OC^2+OD^2=CE^2$ or that $AD^2+CB^2=CE^2$ or that $AD=EB$
$\angle DCB=\widehat{ABE}$ ⊥ each to each $\therefore \angle DAB=\widehat{ABE}$ $\therefore$ arc $AED=BDE$
$\therefore AD=EB$

Ex. 476. If the diameter of a circle is divided into any two segments, and upon these segments as diameters semicircumferences are described upon opposite sides of the diameter, these semicircumferences divide the circle into two parts which have the same ratio as the two segments of the diameter.

Ex. 477. The diagonals that join any vertex of a regular polygon to all the vertices not adjacent divide the angle at that vertex into as many equal parts less two as the polygon has sides.

PROBLEMS OF CONSTRUCTION.

Ex. 478. To circumscribe an equilateral triangle about a given circle.

Ex. 479. To circumscribe a square about a given circle.

Ex. 480. To circumscribe a regular hexagon about a given circle.

Ex. 481. To circumscribe a regular octagon about a given circle.

Ex. 482. To circumscribe a regular pentagon about a given circle.

Ex. 483. To draw through a given point a line so as to divide a given circumference into two parts having the ratio 3 : 7.

Ex. 484. To construct a circumference equal to the sum of two given circumferences.

Ex. 485. To construct a circumference equal to the difference of two given circumferences.

Ex. 486. To construct a circle equivalent to the sum of two given circles.

Ex. 487. To construct a circle equivalent to the difference of two given circles.

Ex. 488. To construct a circle equivalent to three times a given circle.

Ex. 489. To construct a circle equivalent to three fourths of a given circle.

Ex. 490. To construct a circle whose ratio to a given circle shall be equal to the given ratio $m : n$.

Ex. 491. To divide a given circle by a concentric circumference into two equivalent parts.

Ex. 492. To divide a given circle by concentric circumferences into five equivalent parts.

Ex. 493. To construct an angle of 18°; of 36°; of 9°.

Ex. 494. To construct an angle of 12°; of 24°; of 6°.

To construct with a side of a given length:

Ex. 495. An equilateral triangle.

Ex. 496. A square.

Ex. 497. A regular hexagon.

Ex. 498. A regular octagon.

Ex. 499. A regular pentagon.

Ex. 500. A regular decagon.

Ex. 501. A regular dodecagon.

Ex. 502. A regular pentedecagon.

PROBLEMS OF COMPUTATION.

Ex. 503. Find the area of a circle whose radius is 12 inches.

Ex. 504. Find the circumference and the area of a circle whose diameter is 8 feet.

Ex. 505. A regular pentagon is inscribed in a circle whose radius is R. If the length of a side is a, find the apothem.

Ex. 506. A regular polygon is inscribed in a circle whose radius is R. If the length of a side is a, show that the apothem is $\frac{1}{2}\sqrt{4R^2 - a^2}$.

Ex. 507. Find the area of a regular decagon inscribed in a circle whose radius is 16 inches.

Ex. 508. Find the side of a regular dodecagon inscribed in a circle whose radius is 20 inches.

Ex. 509. Find the perimeter of a regular pentagon inscribed in a circle whose radius is 25 feet.

Ex. 510. The length of each side of a park in the shape of a regular decagon is 100 yards. Find the area of the park.

Ex. 511. Find the cost, at \$2 per yard, of building a wall around a cemetery in the shape of a regular hexagon, that contains 16,627.84 square yards.

Ex. 512. The side of an inscribed regular polygon of n sides is 16 feet. Find the side of an inscribed regular polygon of $2n$ sides.

Ex. 513. If the radius of a circle is R, and the side of an inscribed regular polygon is a, show that the side of the similar circumscribed regular polygon is $\dfrac{2aR}{\sqrt{4R^2 - a^2}}$.

Ex. 514. What is the width of the circular ring between two concentric circumferences whose lengths are 650 feet and 425 feet?

Ex. 515. Find the angle subtended at the centre by an arc 5 feet 10 inches long, if the radius of the circle is 9 feet 4 inches.

Ex. 516. The chord of a segment is 10 feet, and the radius of the circle is 16 feet. Find the area of the segment.

Ex. 517. Find the area of a sector, if the angle at the centre is 20°, and the radius of the circle is 20 inches.

Ex. 518. The chord of half an arc is 12 feet, and the radius of the circle is 18 feet. Find the height of the segment subtended by the whole arc.

Ex. 519. Find the side of a square which is equivalent to a circle whose diameter is 35 feet.

Ex. 520. The diameter of a circle is 15 feet. Find the diameter of a circle twice as large. Three times as large.

Ex. 521. Find the radii of the concentric circumferences that divide a circle 11 inches in diameter into five equivalent parts.

Ex. 522. The perimeter of a regular hexagon is 840 feet, and that of a regular octagon is the same. By how many square feet is the octagon larger than the hexagon?

Ex. 523. The diameter of a bicycle wheel is 28 inches. How many revolutions does the wheel make in going 10 miles?

Ex. 524. Find the diameter of a carriage wheel that makes 264 revolutions in going half a mile.

Ex. 525. The sides of three regular octagons are 6 feet, 7 feet, 8 feet, respectively. Find the side of a regular octagon equivalent to the sum of the three given octagons.

Ex. 526. A circular pond 100 yards in diameter is surrounded by a walk 10 feet wide. Find the area of the walk.

Ex. 527. The span (chord) of a bridge in the form of a circular arc is 120 feet, and the highest point of the arch is 15 feet above the piers. Find the radius of the arc. = 112,5

Ex. 528. Three equal circles are described each tangent to the other two. If the common radius is R, find the area contained between the circles.

Ex. 529. Given p, P, the perimeters of regular polygons of n sides inscribed in and circumscribed about a given circle. Find p', P', the perimeters of regular polygons of $2n$ sides inscribed in and circumscribed about the given circle.

Ex. 530. Given the radius R, and the apothem r of an inscribed regular polygon of n sides. Find the radius R' and the apothem r' of an isoperimetrical regular polygon of $2n$ sides.

MISCELLANEOUS EXERCISES.

THEOREMS.

Ex. 531. If two adjacent angles of a quadrilateral are right angles, the bisectors of the other two angles are perpendicular.

Ex. 532. If two opposite angles of a quadrilateral are right angles, the bisectors of the other two angles are parallel.

Ex. 533. The two lines that join the middle points of the opposite sides of a quadrilateral bisect each other.

Ex. 534. The line that joins the feet of the perpendiculars dropped from the extremities of the base of an isosceles triangle to the opposite sides is parallel to the base.

Ex. 535. If AD bisects the angle A of a triangle ABC, and BD bisects the exterior angle CBF, then angle ADB equals one half angle ACB.

Ex. 536. The sum of the acute angles at the vertices of a pentagram (five-pointed star) is equal to two right angles.

Ex. 537. The altitudes AD, BE, CF of the triangle ABC bisect the angles of the triangle DEF.

Circles with AB, BC, AC as diameters will pass through E and D, E and F, D and F, respectively.

Ex. 538. The segments of any straight line intercepted between the circumferences of two concentric circles are equal.

Ex. 539. If a circle is circumscribed about any triangle, the feet of the perpendiculars dropped from any point in the circumference to the sides of the triangle lie in one straight line.

Ex. 540. Two circles are tangent internally at P, and a chord AB of the larger circle touches the smaller circle at C. Prove that PC bisects the angle APB.

Ex. 541. The diagonals of a trapezoid divide each other into segments which are proportional.

Ex. 542. If through a point P in the circumference of a circle two chords are drawn, the chords and the segments between P and a chord parallel to the tangent at P are reciprocally proportional.

Ex. 543. The perpendiculars from two vertices of a triangle upon the opposite sides divide each other into segments reciprocally proportional.

Ex. 544. The perpendicular from any point of a circumference upon a chord is the mean proportional between the perpendiculars from the same point upon the tangents drawn at the extremities of the chord.

Ex. 545. In an isosceles right triangle either leg is the mean proportional between the hypotenuse and the perpendicular upon it from the vertex of the right angle.

Ex. 546. If two circles intersect in the points A and B, and through A any secant CAD is drawn limited by the circumferences at C and D, the straight lines BC, BD are to each other as the diameters of the circles.

$\widehat{BKx} = \angle \widehat{BAx}$, $\angle \widehat{BAx} = \widehat{COB}$, $\therefore \frac{Bx}{BC} = \frac{BK}{BD}$

Ex. 547. The area of a triangle is equal to half the product of its perimeter by the radius of the inscribed circle.

Ex. 548. The perimeter of a triangle is to one side as the perpendicular from the opposite vertex is to the radius of the inscribed circle.

Ex. 549. If three straight lines AA', BB', CC', drawn from the vertices of a triangle ABC to the opposite sides, pass through a common point O within the triangle, then

$$\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1.$$

Ex. 550. ABC is a triangle, M the middle point of AB, P any point in AB between A and M. If MD is drawn parallel to PC, meeting BC at D, the triangle BPD is equivalent to half the triangle ABC.

Ex. 551. Two diagonals of a regular pentagon, not drawn from a common vertex, divide each other in extreme and mean ratio.

Ex. 552. If all the diagonals of a regular pentagon are drawn, another regular pentagon is thereby formed.

Ex. 553. The area of an inscribed regular dodecagon is equal to three times the square of the radius. because it's equal to $\frac{12 \cdot a \cdot a \cdot \sin 30}{2} = 3a^2$

Ex. 554. The area of a square inscribed in a semicircle is equal to two fifths the area of the square inscribed in the circle.

Ex. 555. The area of a circle is greater than the area of any polygon of equal perimeter.

Ex. 556. The circumference of a circle is less than the perimeter of any polygon of equal area.

PROBLEMS OF LOCI.

Ex. 557. Find the locus of the centre of the circle inscribed in a triangle that has a given base and a given angle at the vertex.

Ex. 558. Find the locus of the intersection of the altitudes of a triangle that has a given base and a given angle at the vertex.

Ex. 559. Find the locus of the extremity of a tangent to a given circle, if the length of the tangent is equal to a given line.

Ex. 560. Find the locus of a point, tangents drawn from which to a given circle form a given angle.

Ex. 561. Find the locus of the middle point of a line drawn from a given point to a given straight line.

Ex. 562. Find the locus of the vertex of a triangle that has a given base and a given altitude.

Ex. 563. Find the locus of a point the sum of whose distances from two given parallel lines is equal to a given length.

Ex. 564. Find the locus of a point the difference of whose distances from two given parallel lines is equal to a given length.

Ex. 565. Find the locus of a point the sum of whose distances from two given intersecting lines is equal to a given length.

Ex. 566. Find the locus of a point the difference of whose distances from two given intersecting lines is equal to a given length.

Ex. 567. Find the locus of a point whose distances from two given points are in the given ratio $m : n$.

Ex. 568. Find the locus of a point whose distances from two given parallel lines are in the given ratio $m : n$.

Ex. 569. Find the locus of a point whose distances from two given intersecting lines are in the given ratio $m : n$.

Ex. 570. Find the locus of a point the sum of the squares of whose distances from two given points is constant.

Ex. 571. Find the locus of a point the difference of the squares of whose distances from two given points is constant.

Ex. 572. Find the locus of the vertex of a triangle that has a given base and the other two sides in the given ratio $m : n$.

PROBLEMS OF CONSTRUCTION.

Ex. 573. To divide a given trapezoid into two equivalent parts by a line parallel to the bases.

Ex. 574. To divide a given trapezoid into two equivalent parts by a line through a given point in one of the bases.

Ex. 575. To construct a regular pentagon, given one of the diagonals.

Ex. 576. To divide a given straight line into two segments such that their product shall be the maximum.

Ex. 577. To find a point in a semicircumference such that the sum of its distances from the extremities of the diameter shall be the maximum.

Ex. 578. To draw a common secant to two given circles exterior to each other such that the intercepted chords shall have the given lengths a, b.

Ex. 579. To draw through one of the points of intersection of two intersecting circles a common secant which shall have a given length.

Ex. 580. To construct an isosceles triangle, given the altitude and one of the equal base angles.

Ex. 581. To construct an equilateral triangle, given the altitude.

Ex. 582. To construct a right triangle, given the radius of the inscribed circle and the difference of the acute angles.

Ex. 583. To construct an equilateral triangle so that its vertices shall lie in three given parallel lines.

Ex. 584. To draw a line from a given point to a given straight line which shall be to the perpendicular from the given point as $m : n$.

Ex. 585. To find a point within a given triangle such that the perpendiculars from the point to the three sides shall be as the numbers m, n, p.

Ex. 586. To draw a straight line equidistant from three given points.

Ex. 587. To draw a tangent to a given circle such that the segment intercepted between the point of contact and a given straight line shall have a given length.

Ex. 588. To inscribe a straight line of a given length between two given circumferences and parallel to a given straight line.

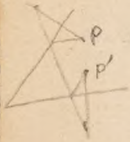
Ex. 589. To draw through a given point a straight line so that its distances from two other given points shall be in a given ratio.

Ex. 590. To construct a square equivalent to the sum of a given triangle and a given parallelogram.

Ex. 591. To construct a rectangle having the difference of its base and altitude equal to a given line, and its area equivalent to the sum of a given triangle and a given pentagon.

Ex. 592. To construct a pentagon similar to a given pentagon and equivalent to a given trapezoid.

Ex. 593. To find a point whose distances from three given straight lines shall be as the numbers m, n, p.



Ex. 594. Given an angle and two points P and P' between the sides of the angle. To find the shortest path from P to P' that shall touch both sides of the angle.

Ex. 595. To construct a triangle, given its angles and its area.

Ex. 596. To transform a given triangle into a triangle similar to another given triangle.

Ex. 597. Given three points A, B, C. To find a fourth point P such that the areas of the triangles APB, APC, BPC shall be equal.

Ex. 598. To construct a triangle, given its base, the ratio of the other sides, and the angle included by them.

Ex. 599. To divide a given circle into n equivalent parts by concentric circumferences.

Ex. 600. In a given equilateral triangle to inscribe three equal circles tangent to each other, each circle tangent to two sides of the triangle.

Ex. 601. Given an angle and a point P between the sides of the angle. To draw through P a straight line that shall form with the sides of the angle a triangle with the perimeter equal to a given length a.

Ex. 602. In a given square to inscribe four equal circles, so that each circle shall be tangent to two of the others and also tangent to two sides of the square.

Ex. 603. In a given square to inscribe four equal circles, so that each circle shall be tangent to two of the others and also tangent to one side of the square.

SOLID GEOMETRY.

Book VI.

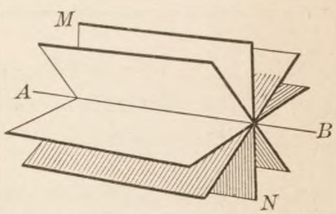
LINES AND PLANES IN SPACE.

DEFINITIONS.

492. Def. A **plane** is a surface such that a straight line joining any two points in it lies wholly in the surface. A plane is understood to be indefinite in extent; but is usually represented by a parallelogram lying in the plane.

493. Def. A plane is said to be **determined** by given lines or points, if no other plane can contain the given lines or points without coinciding with that plane.

494. Cor. 1. *One straight line does not determine a plane.*



For a plane can be made to turn about any straight line AB in it, and thus assume as many different positions as we please.

495. Cor. 2. *A straight line and a point not in the line determine a plane.*

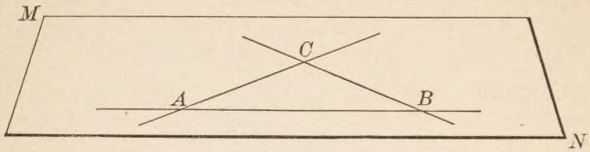
For, if a plane containing a straight line AB and any point C not in AB is made to revolve either way about AB, it will no longer contain the point C.

496. Cor. 3. *Three points not in a straight line determine a plane.*

For by joining two of the points we have a straight line and a point without it, and these determine the plane. § 495

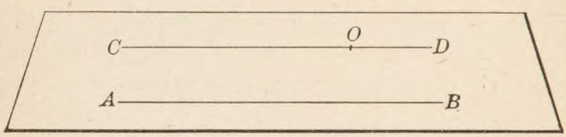
497. Cor. 4. *Two intersecting lines determine a plane.*

For the plane containing one of these lines and any point of the other line not the point of intersection is determined. § 495



498. Cor. 5. *Two parallel lines determine a plane.*

For two parallel lines lie in a plane (§ 103), and a plane containing either parallel and a point in the other is determined. § 495



499. Def. When we suppose a plane to be drawn through given points or lines, we are said to **pass** the plane through the given points or lines.

500. Def. When a straight line is drawn from a point to a plane, its intersection with the plane is called its **foot**.

501. Def. A straight line is **perpendicular** to a plane, if it is perpendicular to every straight line drawn through its foot in the plane; and the plane is perpendicular to the line.

502. Def. A straight line and a plane are **parallel** if they cannot meet, however far both are produced.

503. Def. A straight line neither perpendicular nor parallel to a plane is said to be **oblique** to the plane.

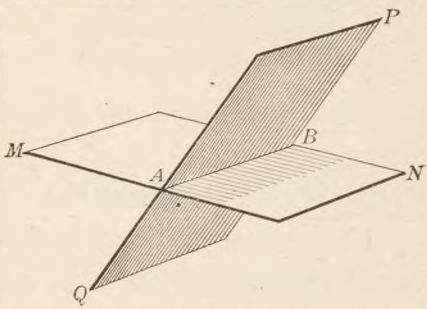
504. Def. Two planes are **parallel** if they cannot meet, however far they are produced.

505. Def. The **intersection of two planes** contains all the points common to the two planes.

LINES AND PLANES.

Proposition I. Theorem.

506. *If two planes cut each other, their intersection is a straight line.*



Let MN and PQ be two planes which cut one another.

To prove that their intersection is a straight line.

Proof. Let A and B be two points common to the two planes.

Draw a straight line through the points A and B.

Then the straight line AB lies in both planes. § 492

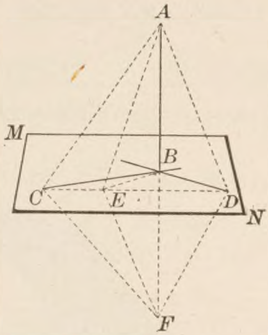
No point not in the line AB can be in both planes; for one plane, and only one, can contain a straight line and a point without the line. § 495

Therefore, the straight line through A and B contains all the points common to the two planes, and is consequently the intersection of the planes. § 505

Q. E. D.

PROPOSITION II. THEOREM.

507. *If a straight line is perpendicular to each of two other straight lines at their point of intersection, it is perpendicular to the plane of the two lines.*



Let AB be perpendicular to BC and BD at B.

To prove that AB is ⊥ to the plane MN of these lines.

Proof. Through B draw in MN *any* other straight line BE, and draw CD cutting BC, BE, BD, at C, E, and D.

Prolong AB to F, making BF equal to AB, and join A and F to each of the points C, E, and D.

Then BC and BD are each ⊥ to AF at its middle point.

$\therefore AC = FC$, and $AD = FD$. § 160

$\therefore \triangle ACD = \triangle FCD$. § 150

$\therefore \angle ACD = \angle FCD$. § 128

That is, $\angle ACE = \angle FCE$.

Hence, the $\triangle$s ACE and FCE are equal. § 143

For $AC = FC$, $CE = CE$, and $\angle ACE = \angle FCE$.

$\therefore AE = FE$; and BE is ⊥ to AF at B. § 161

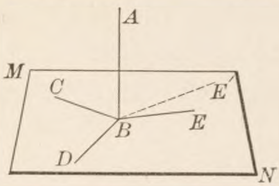
$\therefore AB$ is ⊥ to any and hence every line in MN through B.

$\therefore AB$ is ⊥ to MN. § 501

Q. E. D.

PROPOSITION III. THEOREM.

508. *All the perpendiculars that can be drawn to a straight line at a given point lie in a plane which is perpendicular to the line at the given point.*



Let the plane MN be perpendicular to AB at B.

To prove that BE, any ⊥ to AB at B, lies in MN.

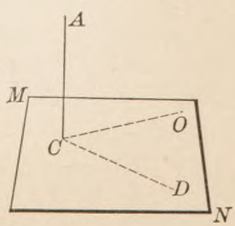
Proof. Let the plane containing AB and BE intersect MN in the line BE'; then AB is ⊥ to BE'. § 501

Since in the plane ABE only one ⊥ can be drawn to AB at B (§ 83), BE and BE' coincide, and BE lies in MN.

Hence, every ⊥ to AB at B lies in the plane MN. Q. E. D.

509. COR. 1. *At a given point in a straight line one plane perpendicular to the line can be drawn, and only one.*

510. COR. 2. *Through a given external point, one plane can be drawn perpendicular to a line, and only one.*



Let AC be the line, and O the point.

Draw OC ⊥ to AC, and CD ⊥ to AC. Then CO and CD determine a plane through O ⊥ to AC.

Only one such plane can be drawn; for only one ⊥ can be drawn to AC from the point O. § 96

Proposition IV. Theorem.

511. *Through a given point there can be one perpendicular to a given plane, and only one.*

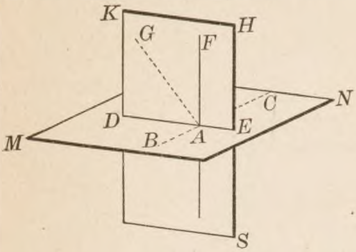


Fig. 1.

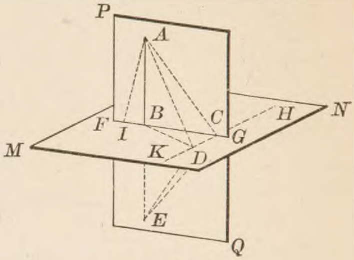


Fig. 2.

Case 1. *When the given point is in the given plane.*

Let A be the given point in the plane MN (Fig. 1).

To prove that there can be one perpendicular to the plane MN at A, and only one.

Proof. Through A draw in MN any line BC, and pass through A a plane $KS \perp$ to BC, cutting MN in DE.

At A erect in the plane KS the line $AF \perp$ to DE.

The line BC, being $\perp$ to the plane KS by construction, is $\perp$ to AF, which passes through its foot in the plane. § 501

That is, AF is $\perp$ to BC; and as it is $\perp$ to DE by construction, it is $\perp$ to the plane MN. § 507

Moreover, any other line AG drawn from A is oblique to MN. For AF and AG intersecting in A determine a plane KS, which cuts MN in the straight line DE; and as AF is $\perp$ to MN, it is $\perp$ to DE (§ 501); hence, AG is oblique to DE (§ 83), and therefore to MN. § 503

Therefore, AF is the only $\perp$ to MN at the point A.

CASE 2. *When the given point is without the given plane.*

Let A be the given point, and MN the given plane (Fig. 2).

To prove that there can be one perpendicular from A to MN, and only one.

Proof. Draw in MN any line HK, and pass through A a plane $PQ \perp$ to HK, cutting MN in FG, and HK in C.

Let fall from A, in the plane PQ, a $\perp$ AB upon FG.

Draw in the plane MN any other line BD from B, intersecting HK in D.

Prolong AB to E, making BE equal to AB,

and join A and E to each of the points C and D.

Since DC is $\perp$ to PQ by construction, and CA and CE lie in PQ, the $\angle$s DCA and DCE are right angles. § 501

The rt. $\triangle$s DCA and DCE are equal. § 144

For DC is common; and $CA = CE$. § 160

$\therefore DA = DE$. § 128

$\therefore BD$ is $\perp$ to AE at B. § 161

That is, AB is $\perp$ to BD, any straight line drawn in MN through its foot, and therefore is $\perp$ to MN. § 501

Moreover, every other straight line AI drawn from A to the plane is oblique to MN. For the lines AB and AI determine a plane PQ which cuts the plane MN in the line FG.

The line AB, being $\perp$ to the plane MN, is $\perp$ to FG. § 501

$\therefore AI$ is oblique to FG, and consequently to MN. § 503

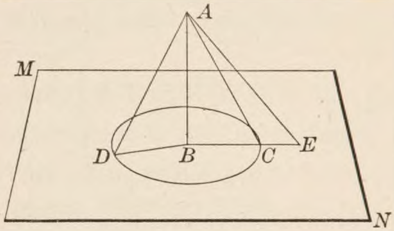
Therefore, AB is the only $\perp$ from A to MN. Q.E.D.

512. COR. *The perpendicular is the shortest line from a point to a plane.*

513. DEF. The **distance from a point to a plane** is the length of the perpendicular from the point to the plane.

Proposition V. Theorem.

514. *Oblique lines drawn from a point to a plane, meeting the plane at equal distances from the foot of the perpendicular, are equal; and of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular the more remote is the greater.*



Let AC and AD cut off the equal distances BC and BD from the foot of the perpendicular AB, and let AD and AE cut off the unequal distances BD and BE, and BE be greater than BD.

To prove that $AC = AD$, *and* $AE > AD$.

Proof. The rt. $\triangle$ ABC and ABD are equal. § 144

For AB is common, and $BC = BD$. Hyp.

$\therefore AC = AD$. § 128

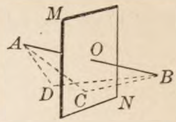
The rt. $\triangle$ ABE, ABC have AB common, and $BE > BC$.

$\therefore AE > AC$ (§ 101), and hence $AE > AD$. Q.E.D.

515. Cor. 1. *Equal oblique lines from a point to a plane meet the plane at equal distances from the foot of the perpendicular; and of two unequal lines the greater meets the plane at the greater distance from the foot of the perpendicular.*

516. Cor. 2. *The locus of a point in space equidistant from all points in the circumference of a circle is a straight line through the centre, perpendicular to the plane of the circle.*

517. Cor 3. *The locus of a point in space equidistant from the extremities of a straight line is the plane perpendicular to this line at its middle point.*

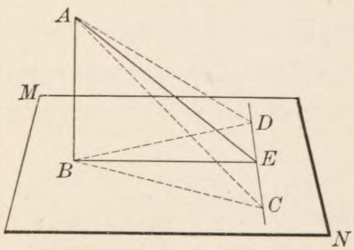


For any point C in this plane lies in a $\perp$ to AB at O, its middle point; hence, CA and CB are equal. § 160

And any point D without the plane MN cannot lie in a $\perp$ to AB at O, and hence is unequally distant from A and B. § 160

Proposition VI. Theorem.

518. *If from the foot of a perpendicular to a plane a straight line is drawn at right angles to any line in the plane, the line drawn from its intersection with the line in the plane to any point of the perpendicular is perpendicular to the line of the plane.*



Let AB be a perpendicular to the plane MN, BE a perpendicular from B to any line CD in MN, and EA any line from E to AB.

To prove that AE *is* $\perp$ *to* CD.

Proof. Take EC and ED equal; draw BC, BD, AC, AD.

Now $BC = BD$. § 95

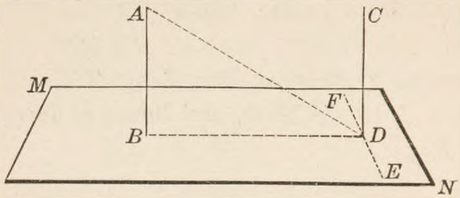
$\therefore AC = AD$. § 514

$\therefore AE$ is $\perp$ to CD. § 161

Q.E.D.

Proposition VII. Theorem.

519. *Two straight lines perpendicular to the same plane are parallel.*



Let AB be perpendicular to MN at B, and CD to MN at D.

To prove that AB and CD are parallel.

Proof. From A any external point in AB draw AD and BD.

Through D draw EF in the plane $MN \perp$ to BD.

Then CD is $\perp$ to EF (§ 501), and AD is $\perp$ to EF. § 518

Therefore, CD, AD, and BD lie in the same plane. § 508

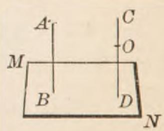
Also the line AB lies in this plane. § 492

But AB and CD are both $\perp$ to BD. § 501

Therefore, AB and CD are parallel. § 104

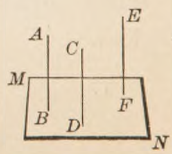
Q. E. D.

520. Cor. 1. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*



For if through any point O of CD a line is drawn $\perp$ to MN, it is $\parallel$ to AB (§ 519), and CD coincides with this $\perp$ and is $\perp$ to MN. § 105

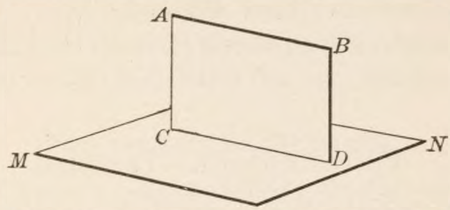
521. Cor. 2. *If two straight lines are parallel to a third straight line, they are parallel to each other.*



For a plane $MN \perp$ to CD is $\perp$ to AB and EF. § 520

Proposition VIII. Theorem.

522. *If two straight lines are parallel, every plane containing one of the lines, and only one, is parallel to the other line.*



Let AB and CD be two parallel lines, and MN any plane containing CD but not AB.

To prove that AB and MN are parallel.

Proof. The lines AB and CD are in the same plane, § 103 and this plane intersects the plane MN in the line CD. Hyp.

Therefore, if AB meets the plane MN at all, the point of meeting must be in the line CD.

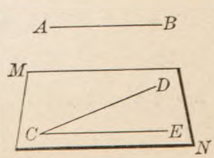
But since AB is $\parallel$ to CD, AB cannot meet CD.

Therefore, AB cannot meet the plane MN.

Hence, AB is $\parallel$ to MN. § 502

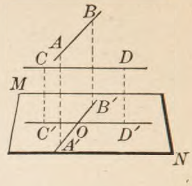
Q. E. D.

523. Cor. 1. *Through either of two straight lines not in the same plane one plane, and only one, can be passed parallel to the other.*



For if AB and CD are the lines, and we pass a plane through CD and the line CE drawn $\parallel$ to AB, the plane MN determined by CD and CE is $\parallel$ to AB. § 522

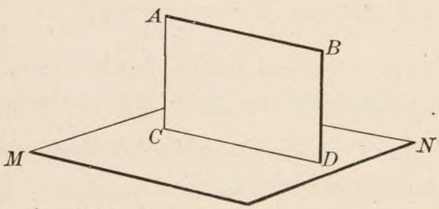
524. Cor. 2. *Through a given point a plane can be passed parallel to any two given straight lines in space.*



For if O is the given point, and AB and CD the given lines, by drawing through O a line $A'B' \parallel$ to AB, and also a line $C'D' \parallel$ to CD, we shall have two lines $A'B'$ and $C'D'$ which determine a plane passing through O and $\parallel$ to each of the lines AB and CD. § 522

Proposition IX. Theorem.

525. *If a straight line is parallel to a plane, the intersection of the plane with any plane passed through the given line is parallel to that line.*



Let the line AB be parallel to the plane MN, and let CD be the intersection of MN with any plane AD passed through AB.

To prove that AB and CD are parallel.

Proof. The lines AB and CD are in the same plane AD.

Since CD lies in the plane MN, if AB meets CD it must meet the plane MN.

But AB is by hypothesis $\parallel$ to MN, and therefore cannot meet it; that is, it cannot meet CD, however far they may be produced.

Hence, AB and CD are parallel. § 103

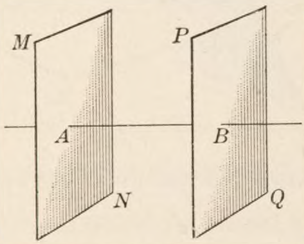
Q. E. D.

526. Cor. *If a given straight line and a plane are parallel, a parallel to the given line drawn through any point of the plane lies in the plane.*

For the plane determined by the given line AB and any point C of the plane cuts MN in a line $CD \parallel$ to AB (§ 525); but through C only one parallel to AB can be drawn (§ 105); therefore, a line drawn through $C \parallel$ to AB coincides with CD, and hence lies in the plane MN.

Proposition X. Theorem.

527. *Two planes perpendicular to the same straight line are parallel.*



Let MN and PQ be two planes perpendicular to the straight line AB.

To prove that MN and PQ are parallel.

Proof. MN and PQ cannot meet.

For if they could meet, we should have two planes from a point of their intersection $\perp$ to the same straight line.

But this is impossible. § 510

Therefore, MN and PQ are parallel. § 504

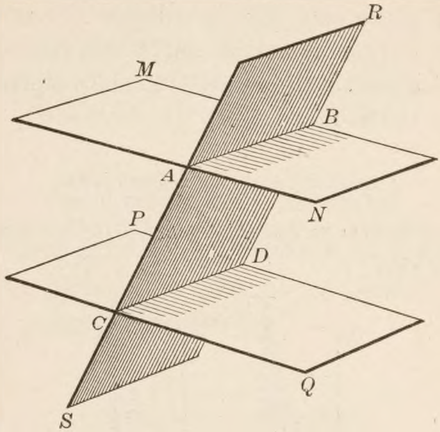
Q.E.D.

Ex. 604. Find the locus of points in space equidistant from two given parallel planes.

Ex. 605. Find the locus of points in space equidistant from two given points and also equidistant from two given parallel planes.

PROPOSITION XI. THEOREM.

528. *The intersections of two parallel planes by a third plane are parallel lines.*



Let the parallel planes MN and PQ be cut by RS.

To prove that the intersections AB and CD are parallel.

Proof. AB and CD are in the same plane RS.

They are also in the parallel planes MN and PQ, which cannot meet, however far they extend. § 504

Therefore, AB and CD cannot meet, and are parallel. § 103

Q. E. D.

529. COR. 1. *Parallel lines included between parallel planes are equal.*

For if the lines AC and BD are parallel, the plane of these lines will intersect MN and PQ in the parallel lines AB and CD. § 528

$\therefore ABDC$ is a parallelogram. § 166

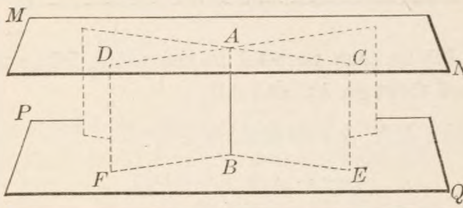
$\therefore AC$ and BD are equal. § 178

530. COR. 2. *Two parallel planes are everywhere equally distant.*

For $\perp$s dropped from *any* points in MN to PQ measure the distances of these points from PQ. But these $\perp$s are parallel (§ 519), and hence equal (§ 529). Therefore, *all* points in MN are equidistant from PQ.

PROPOSITION XII. THEOREM.

531. *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*



Let AB be perpendicular to MN and PQ parallel to MN.

To prove that AB is perpendicular to PQ.

Proof. Pass through the line AB any two planes intersecting MN in the lines AC and AD, and PQ in BE and BF. Then AC and AD are $\parallel$ to BE and BF, respectively. § 528

But AB is $\perp$ to AC and AD. § 501

$\therefore$ AB is $\perp$ to their parallels BE and BF. § 107

Therefore, AB is $\perp$ to PQ. § 507

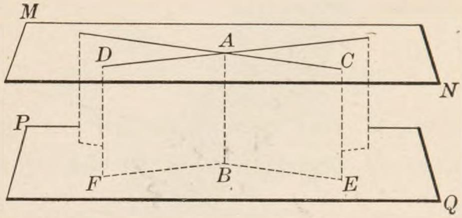
Q. E. D.

532. COR. *Through a given point one plane, and only one, can be drawn parallel to a given plane.*

For if a line is drawn from A $\perp$ to PQ, a plane passing through A $\perp$ to this line is $\parallel$ to PQ (§ 527); and since through a point in a line only one plane can be drawn $\perp$ to the line (§ 509), only one plane can be drawn through A $\parallel$ to PQ.

Proposition XIII. Theorem.

533. *If two intersecting straight lines are each parallel to a plane, the plane of these lines is parallel to that plane.*



Let AC and AD be each parallel to the plane PQ, and let MN be the plane passed through AC and AD.

To prove that MN is parallel to PQ.

Proof. Draw $AB \perp$ to PQ.

Pass a plane through AB and AC intersecting PQ in BE, and a plane through AB and AD intersecting PQ in BF.

Then AB is $\perp$ to BE and BF. § 501

Also, BE is $\parallel$ to AC, and BF is $\parallel$ to AD. § 525

Therefore, AB is $\perp$ to AC and to AD. § 107

Therefore, AB is $\perp$ to the plane MN. § 507

Hence, MN and PQ are parallel. § 527

Q. E. D.

Ex. 606. Find the locus of all lines drawn through a given point, parallel to a given plane.

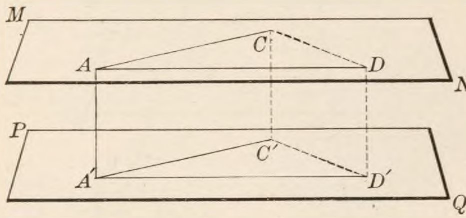
Ex. 607. Find the locus of points in a given plane which are equidistant from two given points not in the plane.

Ex. 608. Find the locus of a point in space equidistant from three given points not in a straight line.

Ex. 609. Find a point in a plane such that the sum of its distances from two given points on the same side of the plane shall be a minimum.

Proposition XIV. Theorem.

534. *If two angles not in the same plane have their sides respectively parallel and lying on the same side of the straight line joining their vertices, they are equal, and their planes are parallel.*



Let the corresponding sides of angles A and A′ in the planes MN and PQ be parallel, and lie on the same side of AA′.

To prove that $\angle A = \angle A'$, *and that* MN *is* $\parallel$ *to* PQ.

Proof. Take AD and $A'D'$ equal, also AC and $A'C'$ equal.

Draw DD', CC', CD, $C'D'$.

Since AD is equal and $\parallel$ to $A'D'$, the figure $ADD'A'$ is a parallelogram, and AA' is equal and $\parallel$ to DD'. § 183

In like manner AA' is equal and $\parallel$ to CC'.

Also, since CC' and DD' are each $\parallel$ to AA', and equal to AA', they are $\parallel$ and equal.

$\therefore CD = C'D'$. § 183

$\therefore \triangle ADC = \triangle A'D'C'$. § 150

$\therefore \angle A = \angle A'$. § 128

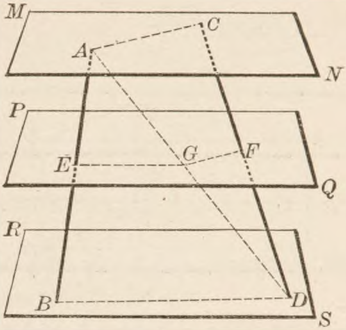
Now PQ is $\parallel$ to each of the lines AC and AD. § 522

Therefore, PQ is $\parallel$ to MN, the plane of these lines. § 533

Q.E.D.

Proposition XV. Theorem.

535. *If two straight lines are intersected by three parallel planes, their corresponding segments are proportional.*



Let AB and CD be intersected by the parallel planes MN, PQ, RS, in the points A, E, B, and C, F, D.

To prove that $AE : EB = CF : FD$.

Proof. Draw AD cutting the plane PQ in G.

Draw AC, BD, EG, and FG.

Then EG is $\parallel$ to BD, and GF is $\parallel$ to AC. § 528

$\therefore AE : EB = AG : GD$, § 342

and $CF : FD = AG : GD$.

$\therefore AE : EB = CF : FD$. Ax. 1

Q. E. D.

Ex. 610. The line AB meets three parallel planes in the points A, E, B; and the line CD meets the same planes in the points C, F, D. If $AE = 6$ inches, $BE = 8$ inches, $CD = 12$ inches, compute CF and FD.

Ex. 611. To draw a perpendicular to a given plane from a given point without the plane.

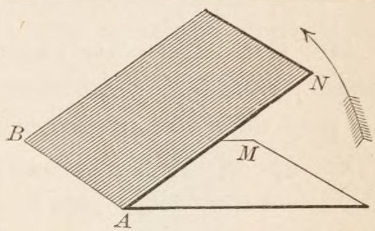
Ex. 612. To erect a perpendicular to a given plane at a given point in the plane.

DIHEDRAL ANGLES.

536. Def. The *opening* between two intersecting planes is called a **dihedral angle.**

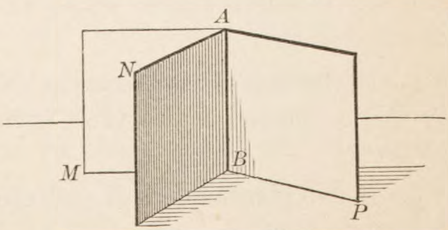
537. Def. The line of intersection *AB* of the planes is the **edge**, the planes *MA* and *NB* are the **faces**, of the dihedral angle.

538. A dihedral angle is designated by its edge, or by its two faces and its edge. Thus, the dihedral angle in the margin may be designated by *AB*, or by *M-AB-N*.



539. In order to have a clear notion of the *magnitude* of the dihedral angle *M-AB-N*, suppose a plane at first in coincidence with the plane *MA* to turn about the edge *AB*, as indicated by the arrow, until it coincides with the plane *NB*. The magnitude of the dihedral angle *M-AB-N* is proportional to the *amount of rotation* of this plane.

540. Def. Two dihedral angles *M-AB-N* and *P-AB-N* are **adjacent** if they have a common edge *AB*, and a common face *NB*, between them.



541. Def. When a plane meets another plane and makes the *adjacent dihedral angles equal*, each of these angles is called a **right dihedral angle.**

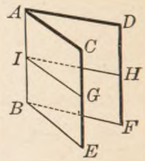
542. Def. A plane is **perpendicular** to another plane if it forms with this second plane a right dihedral angle.

543. Def. Two **vertical dihedral angles** are dihedral angles that have the same edge and the faces of the one are the prolongations of the faces of the other.

544. Def. Dihedral angles are **acute, obtuse, complementary, supplementary**, under the same conditions as plane angles.

545. Def. The **plane angle of a dihedral angle** is the plane angle formed by two straight lines, one in each plane, perpendicular to the edge at the same point.

546. Cor. *The plane angle of a dihedral angle has the same magnitude from whatever point in the edge the perpendiculars are drawn.*



For any two such angles, as CAD, GIH, have their sides respectively parallel (§ 104), and hence are equal. § 534

547. The demonstrations of many properties of dihedral angles are identically the same as the demonstrations of analogous theorems of plane angles.

The following are examples:

1. If a plane meets another plane, it forms with it two adjacent dihedral angles whose sum is equal to two right dihedral angles.

2. If the sum of two adjacent dihedral angles is equal to two right dihedral angles, their exterior faces are in the same plane.

3. If two planes intersect each other, their vertical dihedral angles are equal.

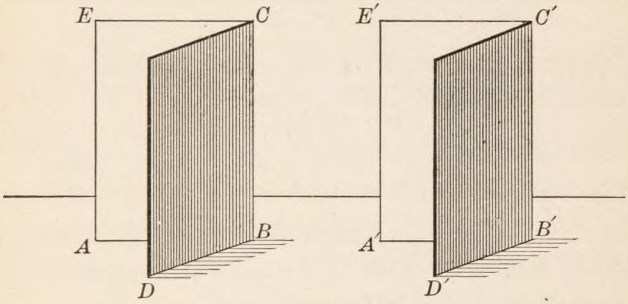
4. If a plane intersects two parallel planes, the alternate-interior dihedral angles are equal; the exterior-interior dihedral angles are equal; the two interior dihedral angles on the same side of the transverse plane are supplementary.

5. When two planes are cut by a third plane, if the alternate-interior dihedral angles are equal, or the exterior-interior dihedral angles are equal, and the edges of the dihedral angles thus formed are parallel, the two planes are parallel.

6. Two dihedral angles whose faces are parallel each to each are either equal or supplementary.

Proposition XVI. Theorem.

548. *Two dihedral angles are equal if their plane angles are equal.*



Let the two plane angles ABD and A′B′D′ of the two dihedral angles D-CB-E and D′-C′B′-E′ be equal.

To prove the dihedral angles D-CB-E and D'-$C'B'$-E' equal.

Proof. Apply D'-$C'B'$-E' to D-CB-E, making the plane angle $A'B'D'$ coincide with its equal ABD.

The line $B'C'$ being $\perp$ to the plane $A'B'D'$ will likewise be $\perp$ to the plane ABD at B, and fall on BC, since at B only one $\perp$ can be erected to this plane. § 511

The two planes $A'B'C'$ and ABC, having in common two intersecting lines AB and BC, coincide. § 497

In like manner the planes $D'B'C'$ and DBC coincide.

Therefore, the two dihedral angles coincide and are equal.

Q. E. D.

PROPOSITION XVII. THEOREM.

549. *Two dihedral angles have the same ratio as their plane angles.*

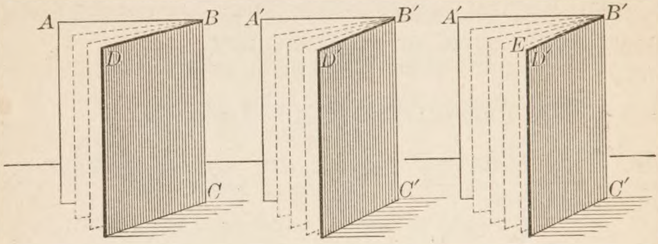


FIG. 1. FIG. 2. FIG. 3.

Let A-BC-D and A′-B′C′-D′ be two dihedral angles, and let their plane angles be ABD and A′B′D′, respectively.

To prove that $A'\text{-}B'C'\text{-}D' : A\text{-}BC\text{-}D = \angle A'B'D' : \angle ABD$.

CASE 1. *When the plane angles are commensurable.*

Proof. Suppose the $\angle s$ ABD and $A'B'D'$ (Figs. 1 and 2) have a common measure, which is contained m times in $\angle ABD$ and n times in $\angle A'B'D'$.

Then $$\angle A'B'D' : \angle ABD = n : m.$$

Apply this measure to $\angle ABD$ and $\angle A'B'D'$, and through the lines of division and the edges BC and $B'C'$ pass planes.

These planes divide $A\text{-}BC\text{-}D$ into m parts, and $A'\text{-}B'C'\text{-}D'$ into n parts, equal each to each. § 548

Therefore, $$A'\text{-}B'C'\text{-}D' : A\text{-}BC\text{-}D = n : m.$$

Therefore, $A'\text{-}B'C'\text{-}D' : A\text{-}BC\text{-}D = \angle A'B'D' : \angle ABD$. Ax. 1

CASE 2. *When the plane angles are incommensurable.*

Proof. Divide the $\angle ABD$ into any number of equal parts, and apply one of these parts to the $\angle A'B'D'$ (Figs. 1 and 3) as a unit of measure.

Since $\angle ABD$ and $\angle A'B'D'$ are incommensurable, a certain number of these parts will form the $\angle A'B'E$, leaving a remainder $\angle EB'D'$, less than one of the parts.

Pass a plane through $B'E$ and $B'C'$.

Since the plane angles of the dihedral angles $A\text{-}BC\text{-}D$ and $A'\text{-}B'C'\text{-}E$ are commensurable,

$$A'\text{-}B'C'\text{-}E : A\text{-}BC\text{-}D = \angle A'B'E : \angle ABD. \qquad \text{Case 1}$$

By increasing the number of equal parts into which $\angle ABD$ is divided, we can diminish at pleasure the magnitude of each part, and therefore make $\angle EB'D'$ less than any assigned value, however small, since $\angle EB'D'$ is always less than one of the equal parts into which $\angle ABD$ is divided.

But we cannot make $\angle EB'D'$ equal to zero, since by hypothesis $\angle ABD$ and $\angle A'B'D'$ are incommensurable. § 269

Therefore, $\angle EB'D'$ approaches zero as a limit, if the number of parts into which $\angle ABD$ is divided is indefinitely increased; and the corresponding dihedral angle $E\text{-}B'C'\text{-}D'$ approaches zero as a limit. § 275

Therefore, $\angle A'B'E$ approaches $\angle A'B'D'$ as a limit, § 271

and $A'\text{-}B'C'\text{-}E$ approaches $A'\text{-}B'C'\text{-}D'$ as a limit.

Hence, $\dfrac{\angle A'B'E}{\angle ABD}$ approaches $\dfrac{\angle A'B'D'}{\angle ABD}$ as a limit, § 280

and $\dfrac{A'\text{-}B'C'\text{-}E}{A\text{-}BC\text{-}D}$ approaches $\dfrac{A'\text{-}B'C'\text{-}D'}{A\text{-}BC\text{-}D}$ as a limit. § 280

But $\dfrac{\angle A'B'E}{\angle ABD}$ is constantly equal to $\dfrac{A'\text{-}B'C'\text{-}E}{A\text{-}BC\text{-}D}$, Case 1

as $\angle EB'D'$ varies in value and approaches zero as a limit.

Therefore, the limits of these variables are equal. § 284

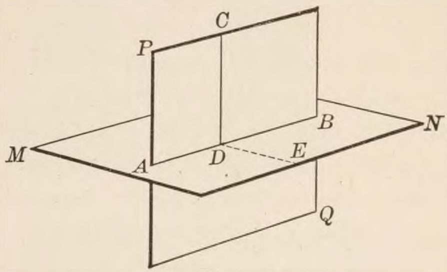
That is,
$$\frac{A'\text{-}B'C'\text{-}D'}{A\text{-}BC\text{-}D} = \frac{\angle A'B'D'}{\angle ABD}.$$

Q. E. D.

550. Cor. *The plane angle of a dihedral angle may be taken as the measure of the dihedral angle.*

PROPOSITION XVIII. THEOREM.

551. *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other plane.*



Let the plane PQ be perpendicular to MN, and let CD be drawn in PQ perpendicular to AB, the intersection of PQ and MN.

To prove that CD is perpendicular to MN.

Proof. In the plane MN draw $DE \perp$ to AB at D.

Then CDE is the measure of the right dihedral angle P-AB-N, and is therefore a right angle. § 550

But, by hypothesis, CDA is a right angle.

Therefore, CD is $\perp$ to DA and DE at their point of intersection, and consequently $\perp$ to their plane MN. § 507

Q. E. D.

552. COR. 1. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection will lie in the other plane.*

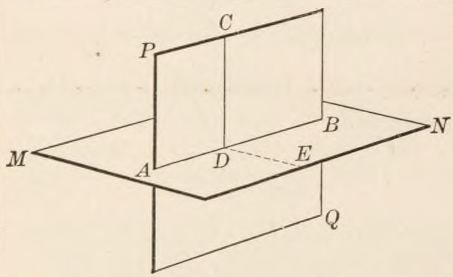
For a line CD drawn in the plane $PAB \perp$ to AB at the point D will be $\perp$ to MN (§ 551). But at the point D only one $\perp$ can be drawn to MN (§ 511). Therefore, a $\perp$ to MN erected at D will coincide with CD and lie in the plane PAB.

553. Cor. 2. *If two planes are perpendicular to each other, a perpendicular to one of them from any point of the other will lie in the other plane.*

For a line CD drawn in the plane PAB from the point $C \perp$ to AB will be $\perp$ to MN (§ 551). But from the point C only one $\perp$ can be drawn to MN (§ 511). Therefore, a $\perp$ to MN drawn from C will coincide with CD and lie in PAB.

Proposition XIX. Theorem.

554. *If a straight line is perpendicular to a plane, every plane passed through this line is perpendicular to the plane.*



Let CD be perpendicular to MN, and PQ be any plane passed through CD intersecting MN in AB.

To prove that PQ is perpendicular to the plane MN.

Proof. Draw DE in the plane $MN \perp$ to AB.

Since CD is $\perp$ to MN, it is $\perp$ to AB. § 501

Therefore, $\angle\ CDE$ is the measure of P-AB-N. § 550

But $\angle\ CDE$ is a right angle. § 501

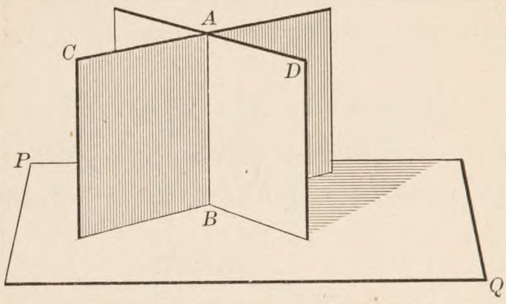
Therefore, PQ is $\perp$ to MN. § 542

Q. E. D.

555. Cor. *A plane perpendicular to the edge of a dihedral angle is perpendicular to each of its faces.*

Proposition XX. Theorem.

556. *If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.*



Let the planes BD and BC intersecting in the line AB be perpendicular to the plane PQ.

To prove that AB is perpendicular to the plane PQ.

Proof. A $\perp$ erected to PQ at B, a point common to the three planes, will lie in the two planes BC and BD. § 552

And since this $\perp$ lies in both the planes BC and BD, it must coincide with their intersection AB. § 506

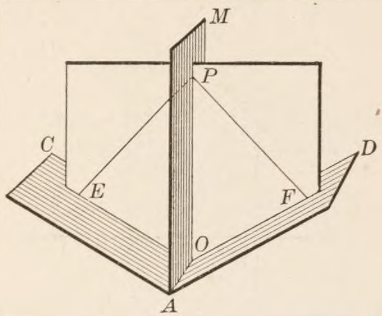
$\therefore$ AB is $\perp$ to the plane PQ. Q.E.D.

557. Cor. 1. *If a plane is perpendicular to each of two intersecting planes, it is perpendicular to their intersection.*

558. Cor. 2. *If a plane is perpendicular to each of two planes that include a right dihedral angle, the intersection of any two of these planes is perpendicular to the third plane, and each of the three intersections is perpendicular to the other two.*

Proposition XXI. Theorem.

559. *Every point in a plane which bisects a dihedral angle is equidistant from the faces of the angle.*



Let the plane AM bisect the dihedral angle formed by the planes AD and AC; and let PE and PF be perpendiculars drawn from any point P in the plane AM to the planes AC and AD.

To prove that $PE = PF$.

Proof. Through PE and PF pass a plane intersecting the planes AC, AD, and AM in the lines OE, OF, and PO.

The plane PEF is $\perp$ to AC and to AD. § 554

Hence, the plane PEF is $\perp$ to their intersection AO. § 557

$\therefore AO$ is $\perp$ to OE, OP, and OF. § 501

$\therefore \angle POE$ is the measure of $M\text{-}AO\text{-}C$, § 550

and $\angle POF$ is the measure of $M\text{-}AO\text{-}D$.

But $M\text{-}AO\text{-}C = M\text{-}AO\text{-}D$. Hyp.

$\therefore \angle POE = \angle POF$.

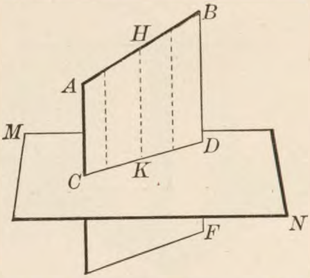
$\therefore$ rt. $\triangle POE =$ rt. $\triangle POF$. § 141

$\therefore PE = PF$. § 128

Q. E. D.

PROPOSITION XXII. THEOREM.

560. *Through a given straight line not perpendicular to a plane, one plane, and only one, can be passed perpendicular to the given plane.*



Let AB be the given line not perpendicular to the plane MN.

To prove that one plane can be passed through AB perpendicular to MN, and only one.

Proof. From any point H of AB draw $HK \perp$ to MN,

and through AB and HK pass a plane AF.

The plane AF is $\perp$ to MN, since it passes through HK, a line $\perp$ to MN. § 554

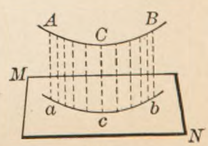
Moreover, if two planes could be passed through $AB \perp$ to the plane MN, their intersection AB would be $\perp$ to MN. § 556

But this is impossible, since AB is by hypothesis not perpendicular to the plane MN.

Hence, through AB only one plane can be passed $\perp$ to MN.

Q. E. D.

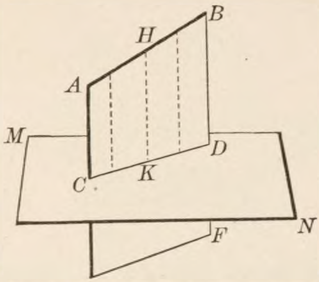
561. DEF. The **projection of a point on a plane** is the foot of the perpendicular from the point to the plane.



562. DEF. The **projection of a line on a plane** is the locus of the projections of its points on the plane.

PROPOSITION XXIII. THEOREM.

563. *The projection of a straight line not perpendicular to a plane upon that plane is a straight line.*



Let AB be the given line, MN the given plane, and CD the projection of AB upon MN.

To prove that CD is a straight line.

Proof. From any point H of AB draw $HK \perp$ to MN, and pass a plane AF through HK and AB.

The plane AF is $\perp$ to MN, § 554

and contains all the $\perp$s drawn from AB to MN. § 553

Hence, CD must be the intersection of these two planes.

Therefore, CD is a straight line. § 506

Q. E. D.

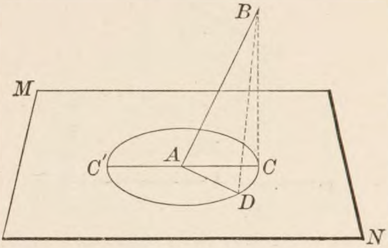
564. COR. *The projection of a straight line perpendicular to a plane upon that plane is a point.*

565. DEF. The plane $ABCD$ is called the **projecting plane** of the line AB upon the plane MN.

566. DEF. The **angle which a line makes with a plane** is the angle which it makes with its projection on the plane; and is called the **inclination** of the line to the plane.

PROPOSITION XXIV. THEOREM.

567. *The acute angle which a straight line makes with its projection upon a plane is the least angle which it makes with any line of the plane.*



Let BA meet the plane MN at A, and let AC be its projection upon the plane MN, and AD any other line drawn through A in the plane.

To prove that $\angle BAC$ *is less than* $\angle BAD$.

Proof. Take AD equal to AC, and draw BD.

In the $\triangle BAC$ and BAD,

$BA = BA$, Iden.

$AC = AD$, Const.

but $BC < BD$. § 512

$\therefore \angle BAC$ is less than $\angle BAD$. § 155

Q. E. D.

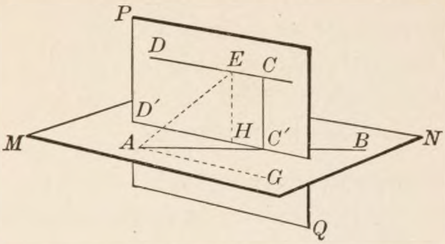
Ex. 613. From a point A, 4 inches from a plane MN, an oblique line AC 5 inches long is drawn to the plane and made to turn around the perpendicular AB dropped from A to the plane. Find the area of the circle described by the point C.

Ex. 614. From a point A, 8 inches from a plane MN, a perpendicular AB is drawn to the plane; with B as centre, and a radius equal to 6 inches, a circle is described in the plane; at any point C of this circumference a tangent CD 24 inches long is drawn. Find the distance from A to D.

Ex. 615. Describe the relative position to a given plane of a line if its projection on the plane is equal to its own length.

Proposition XXV. Theorem.

568. *Between two straight lines not in the same plane, there can be one common perpendicular, and only one.*



Let AB and DC be two lines not in the same plane.

To prove that there can be one common perpendicular between AB and DC, and only one.

Proof. Through any point A of AB draw $AG \parallel$ to DC, and let MN be the plane determined by AB and AG.

Since AG is $\parallel$ to DC, MN is $\parallel$ to DC. § 522

Through DC pass the plane $PQ \perp$ to MN, intersecting the plane MN in $D'C'$. Then $D'C'$ is $\parallel$ to DC. § 525

$D'C'$ must cut AB at some point C', otherwise AB would be $\parallel$ to $D'C'$ (§ 103), and hence $\parallel$ to DC (§ 521). But this is impossible; for AB and DC are not in the same plane. Hyp.

Draw $C'C \perp$ to MN. Then $C'C$ is $\perp$ to AB. § 501

But $C'C$ is in the plane PQ (§ 552) and is $\perp$ to $D'C'$. § 501

$\therefore C'C$ is $\perp$ to DC. § 107

$\therefore C'C$ is $\perp$ to AB and DC.

Again, $C'C$ is the only $\perp$ to both AB and DC. For, if possible, let EA be any other line $\perp$ to AB and DC. Then EA is $\perp$ to AG (§ 107), and hence $\perp$ to MN. § 507

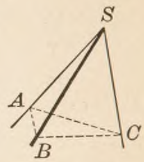
Draw $EH \perp$ to $D'C'$. Then EH is $\perp$ to MN (§ 551), and we have two $\perp$s from E to MN. But this is impossible. § 511

Hence, $C'C$ is the only common $\perp$ to DC and AB. Q. E. D.

POLYHEDRAL ANGLES.

569. Def. The *opening* of three or more planes which meet at a common point is called a **polyhedral angle.**

570. Def. The common point S is the **vertex** of the angle, and the intersections of the planes SA, SB, etc., are its **edges**; the portions of the planes included between the edges are its **faces**, and the angles formed by the edges are its **face angles.**



571. The *magnitude* of a polyhedral angle depends upon the *relative position* of its faces, and not upon their extent.

572. In a polyhedral angle, every two adjacent edges form a face angle, and every two adjacent faces form a dihedral angle. These face angles and dihedral angles are the *parts* of the polyhedral angle.

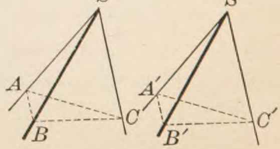
573. Def. A polyhedral angle is **convex**, if every section made by a plane that cuts all its edges is a convex polygon.

574. Def. A polyhedral angle is called **trihedral, tetrahedral,** etc., according as it has *three* faces, *four* faces, etc.

575. Def. A trihedral angle is called **rectangular, bi-rectangular, tri-rectangular,** according as it has *one, two,* or *three* right dihedral angles.

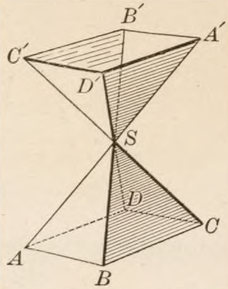
576. Def. A trihedral angle is called **isosceles** if it has two of its face angles equal.

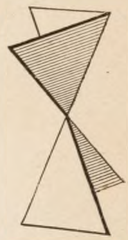
577. Def. Two polyhedral angles can be made to coincide and are **equal** if their corresponding parts are equal and arranged in the **same order.**



578. A polyhedral angle is designated by its vertex, or by its vertex and all the faces taken in order. Thus the polyhedral angle in the margin may be designated by S, or by $S\text{-}ABCD$.

579. If the faces of a polyhedral angle $S\text{-}ABCD$ are produced through the vertex S, another polyhedral angle $S\text{-}A'B'C'D'$ is formed, **symmetrical** with respect to $S\text{-}ABCD$. The face angles ASB, BSC, etc., are equal, respectively, to the face angles $A'SB'$, $B'SC'$, etc. § 93





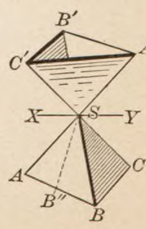
Also the dihedral angles SA, SB, etc., are equal, respectively, to the dihedral angles SA', SB', etc. § 547

(The second figure shows a pair of vertical dihedral angles.)

The edges of $S\text{-}ABCD$ are arranged from **left to right** (counter clockwise) in the order SA, SB, SC, SD, but the edges of $S\text{-}A'B'C'D'$ are arranged from **right to left** (clockwise) in the order SA', SB', SC', SD'; that is, in an order the reverse of the order of the edges in $S\text{-}ABCD$.

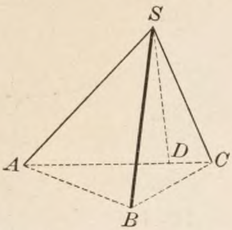
Two symmetrical polyhedral angles, therefore, have all their parts equal, each to each, but arranged in reverse order.

In general, two symmetrical polyhedral angles are not superposable. Thus, if the trihedral angle $S\text{-}A'B'C'$ is made to turn 180° about XY, the bisector of the angle $A'SC$, then SA' will coincide with SC, SC' with SA, and the face $A'SC'$ with ASC; but the dihedral angle SA, and hence the dihedral angle SA', not being equal to SC, the plane $A'SB'$ will not coincide with BSC; and, for a similar reason, the plane $C'SB'$ will not coincide with ASB. Hence, the edge SB' takes some position SB'' not coincident with SB; that is, the trihedral angles are not superposable.



PROPOSITION XXVI. THEOREM.

580. *The sum of any two face angles of a trihedral angle is greater than the third face angle.*



In the trihedral angle S-ABC, let the angle ASC be greater than ASB or BSC.

To prove $\angle ASB + \angle BSC$ *greater than* $\angle ASC$.

Proof. In ASC draw SD, making $\angle ASD$ equal to $\angle ASB$.

Through any point D of SD draw ADC in the plane ASC.

Take SB equal to SD.

Pass a plane through the line AC and the point B.

The $\triangle$ ASD and ASB are equal. § 143

For $AS = AS$, $SD = SB$, and $\angle ASD = \angle ASB$.

$\therefore AD = AB$. § 128

In the $\triangle ABC$, $AB + BC > AC$. § 138

But $AB = AD$.

By subtraction, $BC > DC$. Ax. 5

In the $\triangle$ BSC and DSC,

$SC = SC$, and $SB = SD$, but $BC > DC$.

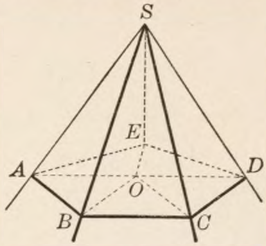
Therefore, $\angle BSC$ is greater than $\angle DSC$. § 155

$\therefore \angle ASB + \angle BSC$ is greater than $\angle ASD + \angle DSC$.

That is, $\angle ASB + \angle BSC$ is greater than $\angle ASC$. Q. E. D.

Proposition XXVII. Theorem.

581. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Let S be a convex polyhedral angle, and let all its edges be cut by a plane, making the section ABCDE.

To prove $\angle ASB + \angle BSC$, *etc., less than four rt.* $\angle$s.

Proof. From any point O within the polygon draw OA, OB, OC, OD, OE.

The number of the $\triangle$s having the common vertex O is the same as the number having the common vertex S.

Therefore, the sum of the $\angle$s of all the $\triangle$s having the common vertex S is equal to the sum of the $\angle$s of all the $\triangle$s having the common vertex O.

But in the trihedral $\angle$s formed at A, B, C, etc.,

$\angle SAE + \angle SAB$ is greater than $\angle EAB$,

$\angle SBA + \angle SBC$ is greater than $\angle ABC$, etc. § 580

Hence, the sum of the $\angle$s at the bases of the $\triangle$s whose common vertex is S is greater than the sum of the $\angle$s at the bases of the $\triangle$s whose common vertex is O. Ax. 4

Therefore, the sum of the $\angle$s at the vertex S is less than the sum of the $\angle$s at the vertex O. Ax. 5

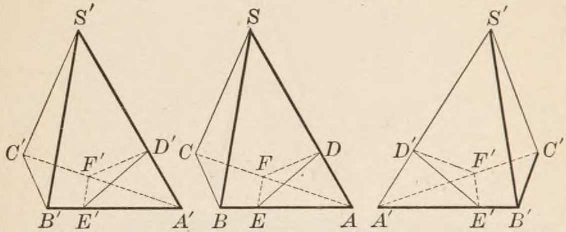
But the sum of the $\angle$s at O is equal to 4 rt. $\angle$s. § 88

Therefore, the sum of the $\angle$s at S is less than 4 rt. $\angle$s.

Q. E. D.

PROPOSITION XXVIII. THEOREM.

582. *Two trihedral angles are equal or symmetrical when the three face angles of the one are respectively equal to the three face angles of the other.*



In the trihedral angles S and S′, let the angles ASB, ASC, BSC be equal to the angles A′S′B′, A′S′C′, B′S′C′, respectively.

To prove that S and S' are equal or symmetrical.

Proof. On the edges of these angles take the six equal distances SA, SB, SC, $S'A'$, $S'B'$, $S'C'$.

Draw AB, BC, AC, $A'B'$, $B'C'$, $A'C'$.

The isosceles △ SAB, SAC, SBC are equal, respectively, to the isosceles △ $S'A'B'$, $S'A'C'$, $S'B'C'$. § 143

$\therefore AB$, BC, CA are equal, respectively, to $A'B'$, $B'C'$, $C'A'$.

$\therefore \triangle ABC = \triangle A'B'C'$. § 150

At any point D in SA draw DE in the face ASB and DF in the face $ASC \perp$ to SA.

These lines meet AB and AC, respectively,

(*since the* ∠ *SAB and SAC are acute, each being one of the equal* ∠ *of an isosceles* △).

Draw EF.

On $A'S'$ take $A'D'$ equal to AD.

Draw $D'E'$ in the face $A'S'B'$ and $D'F'$ in the face $A'S'C' \perp$ to $S'A'$, and draw $E'F'$.

The rt. $\triangle$ ADE and $A'D'E'$ are equal. § 142

For $AD = A'D'$, Const.

and $\angle DAE = \angle D'A'E'$. § 128

$\therefore AE = A'E'$, and $DE = D'E'$. § 128

In like manner $AF = A'F'$, and $DF = D'F'$.

$\therefore \triangle AEF = \triangle A'E'F'$. § 143

For $AE = A'E'$, $AF = A'F'$, and $\angle EAF = \angle E'A'F'$. § 128

$\therefore EF = E'F'$. § 128

$\therefore \triangle EDF = \triangle E'D'F'$. § 150

For $ED = E'D'$, $DF = D'F'$, and $EF = E'F'$.

$\therefore \angle EDF = \angle E'D'F'$. § 128

$\therefore$ the angle $B\text{-}AS\text{-}C = B'\text{-}A'S'\text{-}C'$,

(*since* $\angle$ *EDF and* $E'D'F'$, *the measures of these dihedral* $\angle$, *are equal*).

In like manner it may be proved that the dihedral angles $A\text{-}BS\text{-}C$ and $A\text{-}CS\text{-}B$ are equal, respectively, to the dihedral angles $A'\text{-}B'S'\text{-}C'$ and $A'\text{-}C'S'\text{-}B'$.

$\therefore S$ and S' are equal or symmetrical. §§ 577, 579

Q.E.D.

This demonstration applies to either of the two figures denoted by $S'\text{-}A'B'C'$, which are symmetrical with respect to each other. If the first of these figures is taken, S and S' are equal. If the second is taken, S and S' are symmetrical.

583. Cor. *If two trihedral angles have three face angles of the one equal to three face angles of the other, then the dihedral angles of the one are respectively equal to the dihedral angles of the other.*

EXERCISES.

Ex. 616. Find the locus of points in space equidistant from two given intersecting lines.

Ex. 617. Find the locus of points in space equidistant from all points in the circumference of a circle.

Ex. 618. Find the locus of points in a plane equidistant from a given point without the plane.

Ex. 619. Find a point at equal distances from four points not all in the same plane.

Ex. 620. Two dihedral angles which have their edges parallel and their faces perpendicular are equal or supplementary.

Ex. 621. The projections on a plane of equal and parallel lines are equal and parallel.

Ex. 622. If two face angles of a trihedral angle are equal, the dihedral angles opposite them are equal.

Ex. 623. The planes that bisect the dihedral angles of a trihedral angle intersect in the same straight line.

Ex. 624. If the face angle ASB of the trihedral angle S-ABC is bisected by the line SD, the angle CSD is less than half the sum of the angles ASC and BSC.

Ex. 625. An isosceles trihedral angle and its symmetrical trihedral angle are superposable.

Ex. 626. Find the locus of points equidistant from the three edges of a trihedral angle.

Ex. 627. Find the locus of points equidistant from the three faces of a trihedral angle.

Ex. 628. Two trihedral angles are equal when two dihedral angles and the included face angle of the one are equal, respectively, to two dihedral angles and the included face angle of the other and similarly placed.

Ex. 629. Two trihedral angles are equal when two face angles and the included dihedral angle of the one are equal, respectively, to two face angles and the included dihedral angle of the other and similarly placed.

Book VII.

POLYHEDRONS, CYLINDERS, AND CONES.

POLYHEDRONS.

584. Def. A **polyhedron** is a solid bounded by planes.

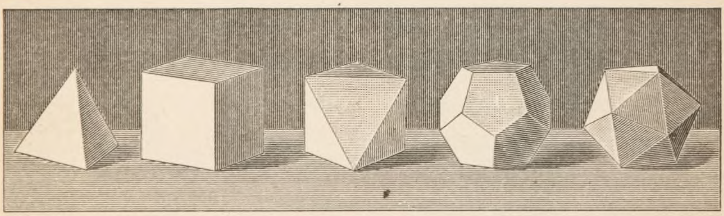
The bounding planes are called the **faces**, the intersections of the faces, the **edges**, and the intersections of the edges, the **vertices**, of the polyhedron.

585. Def. A **diagonal** of a polyhedron is a straight line joining any two vertices not in the same face.

586. Def. A **section** of a polyhedron is the figure formed by its intersection with a plane passing through it.

587. Def. A polyhedron is **convex** if every section is a convex polygon.

Only convex polyhedrons are considered in this work.



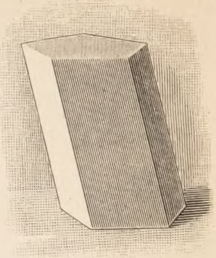
Tetrahedron. Hexahedron. Octahedron. Dodecahedron. Icosahedron.

588. Def. A polyhedron of four faces is called a **tetrahedron**; one of six faces, a **hexahedron**; one of eight faces, an **octahedron**; one of twelve faces, a **dodecahedron**; one of twenty faces, an **icosahedron**.

Note. Full lines in the figures of solids represent visible lines, dashed lines represent invisible lines.

PRISMS AND PARALLELOPIPEDS.

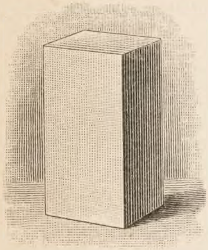
589. Def. A **prism** is a polyhedron of which two faces are equal polygons in parallel planes, and the other faces are parallelograms.



Prism.

The equal polygons are called the **bases** of the prism, the parallelograms, the **lateral faces**, and the intersections of the lateral faces, the **lateral edges** of the prism.

The sum of the areas of the lateral faces of a prism is called its **lateral area**.



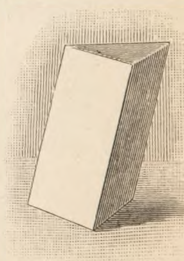
Right Prism.

590. Def. The **altitude** of a prism is the perpendicular distance between the planes of its bases.

591. Def. A **right prism** is a prism whose lateral edges are perpendicular to its bases.

592. Def. A **regular prism** is a right prism whose bases are regular polygons.

593. Def. An **oblique prism** is a prism whose lateral edges are oblique to its bases.

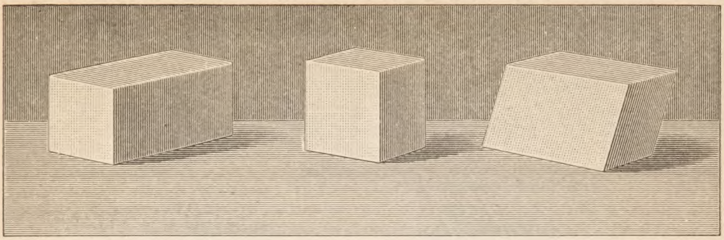


Triangular Prism.

594. The lateral edges of a prism are equal. The lateral edges of a right prism are equal to the altitude.

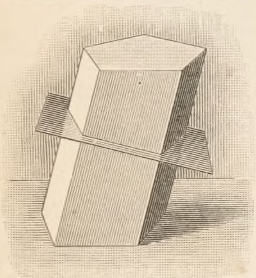
595. Def. Prisms are called **triangular**, **quadrangular**, etc., according as their bases are triangles, quadrilaterals, etc.

596. Def. A **parallelopiped** is a prism whose bases are parallelograms.



Rectangular Parallelopiped. Cube. Oblique Parallelopiped.

597. Def. A **right parallelopiped** is a parallelopiped whose lateral edges are perpendicular to the bases.



Right Section of a Prism.

598. Def. A **rectangular parallelopiped** is a parallelopiped whose six faces are all rectangles.

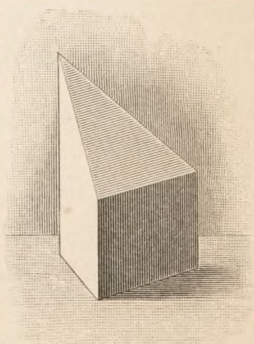
599. Def. A **cube** is a parallelopiped whose six faces are all squares.

600. Def. A cube whose edges are equal to the linear unit is taken as the **unit of volume.**

601. Def. The **volume** of any solid is the number of *units of volume* which it contains.

602. Def. Two solids are **equivalent** if their volumes are equal.

603. Def. A **right section** of a prism is a section made by a plane perpendicular to the lateral edges of the prism.

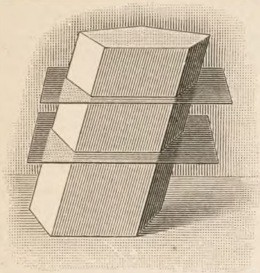


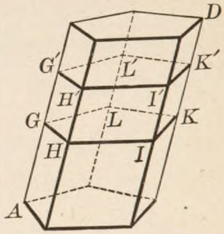
Truncated Prism.

604. Def. A **truncated prism** is the part of a prism included between the base and a section made by a plane oblique to the base.

PROPOSITION I. THEOREM.

605. *The sections of a prism made by parallel planes cutting all the lateral edges are equal polygons.*





Let the prism AD be intersected by parallel planes cutting all the lateral edges, making the sections GK, G'K'.

To prove that $GK = G'K'$.

Proof. The sides GH, HI, IK, etc., are parallel, respectively, to the sides $G'H'$, $H'I'$, $I'K'$, etc. § 528

The sides GH, HI, IK, etc., are equal, respectively, to $G'H'$, $H'I'$, $I'K'$, etc. § 180

The $\angle$s GHI, HIK, etc., are equal, respectively, to $\angle$s $G'H'I'$, $H'I'K'$, etc. § 534

Therefore, $GK = G'K'$. § 203

Q. E. D.

606. COR. *Every section of a prism made by a plane parallel to the base is equal to the base; and all right sections of a prism are equal.*

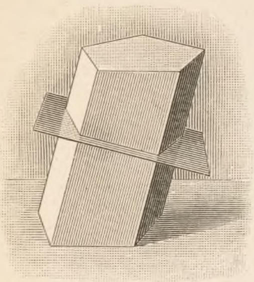
Ex. 630. The diagonals of a parallelopiped bisect one another.

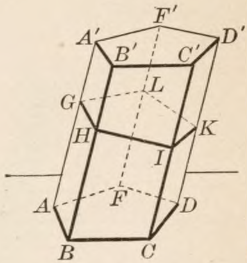
Ex. 631. The lateral faces of a right prism are rectangles.

Ex. 632. Every section of a prism made by a plane parallel to the lateral edges is a parallelogram.

PROPOSITION II. THEOREM.

607. *The lateral area of a prism is equal to the product of a lateral edge by the perimeter of the right section.*





Let GHIKL be a right section of the prism AD′, S its lateral area, E a lateral edge, and P the perimeter of the right section.

To prove that $S = E \times P$.

Proof. $AA' = BB' = CC' = DD' \cdots = E$. § 594

GH is $\perp$ to BB', HI to CC', IK to DD', etc. § 603

$\therefore$ the area of $\square\, AB' = BB' \times GH = E \times GH$, § 400

the area of $\square\, BC' = CC' \times HI = E \times HI$, and so on.

Therefore, S, the sum of these parallelograms, is equal to

$$E(GH + HI + IK + \text{etc.}).$$

But $GH + HI + IK + \text{etc.} = P$.

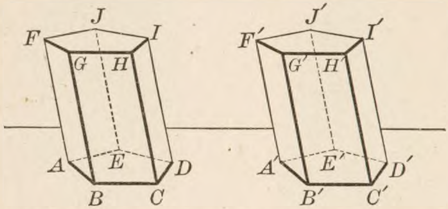
Therefore, $S = E \times P$. Q. E. D.

608. COR. *The lateral area of a right prism is equal to the product of the altitude by the perimeter of the base.*

Ex. 633. Find the lateral area of a right prism, if its altitude is 18 inches and the perimeter of its base 29 inches.

PROPOSITION III. THEOREM.

609. *Two prisms are equal if three faces including a trihedral angle of the one are respectively equal to three faces including a trihedral angle of the other, and are similarly placed.*



In the prisms AI and A′I′, let the faces AD, AG, AJ be respectively equal to A′D′, A′G′, A′J′, and similarly placed.

To prove that $AI = A'I'$.

Proof. The face $\measuredangle s$ BAE, BAF, EAF are equal to the face $\measuredangle s$ $B'A'E'$, $B'A'F'$, $E'A'F'$, respectively. § 203

Therefore, the trihedral angles A and A' are equal. § 582

Apply the trihedral angle A to its equal A'.

Then the face AD coincides with $A'D'$, AG with $A'G'$, and AJ with $A'J'$; and C falls at C', and D at D'.

The lateral edges of the prisms are parallel. § 589

Therefore, CH falls along $C'H'$, and DI along $D'I'$. § 105

Since the points F, G, and J coincide with F', G', and J', each to each, the planes of the upper bases coincide. § 496

Hence, H coincides with H', and I with I'.

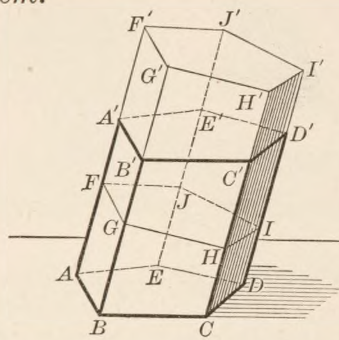
Therefore, the prisms coincide and are equal. Q.E.D.

610. COR. 1. *Two truncated prisms are equal under the hypothesis given in* § 609.

611. COR. 2. *Two right prisms having equal bases and equal altitudes are equal.*

PROPOSITION IV. THEOREM.

612. *An oblique prism is equivalent to a right prism whose base is equal to a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*



Let FI be a right section of the oblique prism AD′, and FI′ a right prism whose lateral edges are equal to the lateral edges of AD′.

To prove that $AD' \approx FI'$.

Proof. If from the equal lateral edges of AD' and FI' we take the lateral edges of FD', which are common to both, the remainders AF and $A'F'$, BG and $B'G'$, etc., are equal. Ax. 3

The upper bases FI and $F'I'$ are equal. § 589

Place AI on $A'I'$ so that FI shall coincide with $F'I'$.

Then FA, GB, etc., coincide with $F'A'$, $G'B'$, etc. § 511

Hence, the faces GA and $G'A'$, HB and $H'B'$, coincide.

But the faces FI and $F'I'$ coincide.

$\therefore$ the truncated prisms AI and $A'I'$ are equal. § 610

Now $\qquad AI + FD' = AD'$, Ax. 9

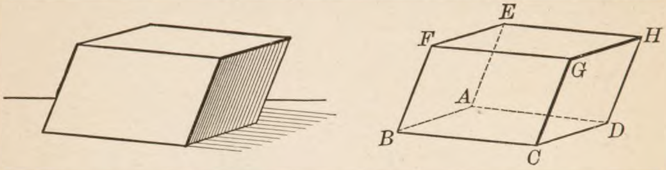
and $\qquad A'I' + FD' = FI'$.

Therefore, $\qquad AD' \approx FI'$. Ax. 2

Q. E. D.

Proposition V. Theorem.

613. *The opposite lateral faces of a parallelopiped are equal and parallel.*



Let BH be a parallelopiped with bases BD and FH.

To prove that the opposite faces BG and AH are equal and parallel.

Proof. BC is equal and parallel to AD, §§ 178, 166

and BF is equal and parallel to AE.

$\therefore \angle FBC = \angle EAD$. § 534

$\therefore BG$ is $\parallel$ to AH. § 534

$\therefore BG = AH$. § 185

In like manner it may be proved that the face BE is equal and parallel to CH. Q. E. D.

614. Cor. *Any two opposite faces of a parallelopiped may be taken as bases.*

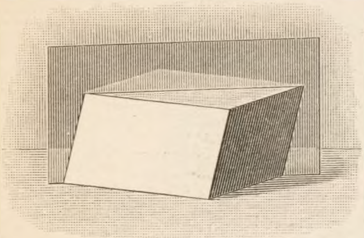
Ex. 634. Find the lateral area of a right prism, if its altitude is 20 inches, and its base is a triangle whose sides are 7 inches, 8 inches, and 9 inches, respectively.

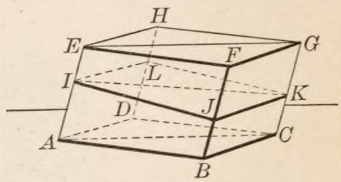
Ex. 635. Find the lateral area of a triangular prism, if its lateral edge is 20 inches, and its right section is a triangle whose sides are 9 inches, 10 inches, and 12 inches, respectively.

Ex. 636. Find the total area of a right prism, if its altitude is 32 inches, and its base is a triangle whose sides are 12 inches, 14 inches, and 16 inches, respectively.

Proposition VI. Theorem.

615. *The plane passed through two diagonally opposite edges of a parallelopiped divides the parallelopiped into two equivalent triangular prisms.*





Let the plane ACGE pass through the opposite edges AE and CG of the parallelopiped AG.

To prove that the parallelopiped AG is divided into two equivalent triangular prisms ABC-F and ADC-H.

Proof. Let $IJKL$ be a right section of the parallelopiped.

The opposite faces AF and DG, and AH and BG, are parallel and equal. § 613

$\therefore IJ$ is $\parallel$ to LK, and IL to JK. § 528

Therefore, $IJKL$ is a parallelogram. § 166

The intersection IK of the right section with the plane $ACGE$ is the diagonal of the $\square$ $IJKL$.

$\therefore \triangle IJK = \triangle ILK$. § 179

But the prism $ABC\text{-}F$ is equivalent to a right prism whose base is IJK and altitude AE, and the prism $ACD\text{-}H$ is equivalent to a right prism whose base is ILK, and altitude AE. § 612

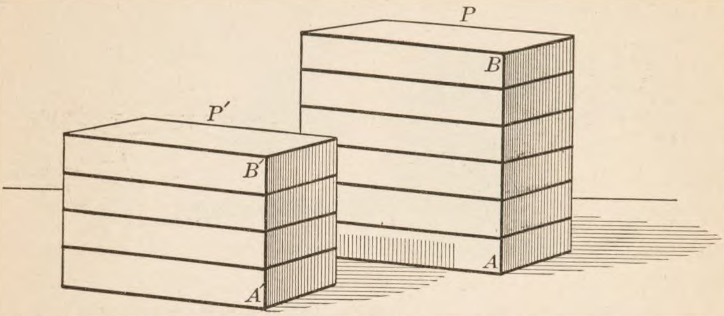
But these two right prisms are equal. § 611

$\therefore ABC\text{-}F \approx ADC\text{-}H$. Ax. 1

Q. E. D.

PROPOSITION VII. THEOREM.

616. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*



Let AB and A′B′ be the altitudes of the two rectangular parallelopipeds P and P′, which have equal bases.

To prove that $P : P' = AB : A'B'$.

CASE 1. *When AB and $A'B'$ are commensurable.*

Proof. Find a common measure of AB and $A'B'$.

Apply this common measure to AB and $A'B'$ as a unit of measure.

Suppose this common measure to be contained m times in AB, and n times in $A'B'$.

Then $AB : A'B' = m : n$.

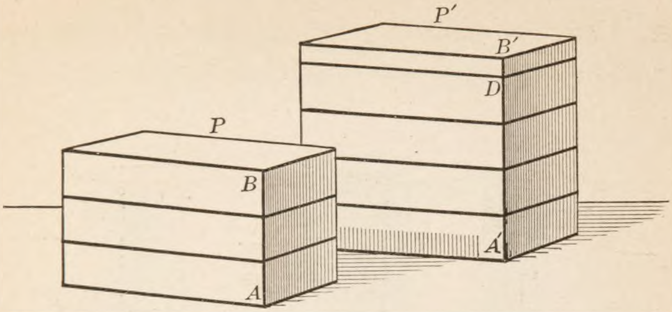
At the several points of division on AB and $A'B'$ pass planes perpendicular to these lines.

The parallelopiped P is divided into m parallelopipeds, and P' into n parallelopipeds, equal each to each. § 611

Therefore, $P : P' = m : n$.

Therefore, $P : P' = AB : A'B'$. Ax. 1

CASE 2. *When AB and $A'B'$ are incommensurable.*



Proof. Let AB be divided into any number of equal parts, and let one of these parts be applied to $A'B'$ as a unit of measure as many times as $A'B'$ will contain it.

Since AB and $A'B'$ are incommensurable, a certain number of these parts will extend from A' to a point D, leaving a remainder DB' less than one of the parts.

Through D pass a plane $\perp$ to $A'B'$, and let Q denote the parallelopiped whose base is the same as that of P', and whose altitude is $A'D$.

Then $Q : P = A'D : AB$. Case 1

If the number of parts into which AB is divided is indefinitely increased, the ratio $Q : P$ approaches $P' : P$ as a limit, and the ratio $A'D : AB$ approaches $A'B' : AB$ as a limit.

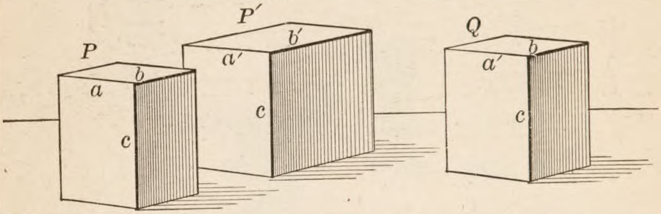
The theorem can be proved for this case by the Method of Limits in the manner shown in § 549. Q.E.D.

617. DEF. The three edges of a rectangular parallelopiped which meet at a common vertex are called its **dimensions**.

618. COR. *Two rectangular parallelopipeds which have two dimensions in common are to each other as their third dimensions.*

PROPOSITION VIII. THEOREM.

619. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*



Let a, b, c, and a′, b′, c, be the three dimensions, respectively, of the two rectangular parallelopipeds P and P′.

To prove that $$\frac{P}{P'} = \frac{a \times b}{a' \times b'}.$$

Proof. Let Q be a third rectangular parallelopiped whose dimensions are a', b, and c.

Now Q has the two dimensions b and c in common with P, and the two dimensions a' and c in common with P'.

Then $$\frac{P}{Q} = \frac{a}{a'},$$

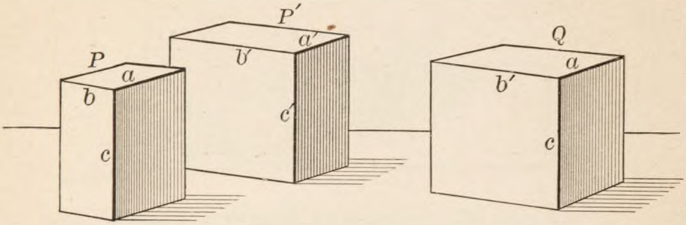
and $$\frac{Q}{P'} = \frac{b}{b'}. \qquad \S\ 618$$

The products of the corresponding members of these two equalities give
$$\frac{P}{P'} = \frac{a \times b}{a' \times b'}.$$
Q.E.D.

620. COR. *Two rectangular parallelopipeds which have one dimension in common are to each other as the products of their other two dimensions.*

Proposition IX. Theorem.

621. *Two rectangular parallelopipeds are to each other as the products of their three dimensions.*



Let a, b, c, and a′, b′, c′, be the three dimensions, respectively, of the two rectangular parallelopipeds P and P′.

To prove that $$\frac{P}{P'} = \frac{a \times b \times c}{a' \times b' \times c'}.$$

Proof. Let Q be a third rectangular parallelopiped whose dimensions are a, b', and c.

Then $$\frac{P}{Q} = \frac{b}{b'},$$ § 618

and $$\frac{Q}{P'} = \frac{a \times c}{a' \times c'}.$$ § 620

The products of the corresponding members of these equalities give

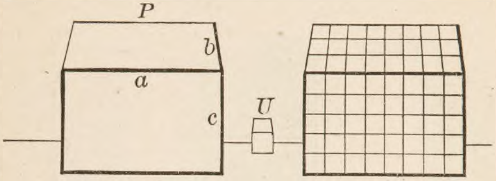
$$\frac{P}{P'} = \frac{a \times b \times c}{a' \times b' \times c'}.$$ Q. E. D.

Ex. 637. Find the ratio of two rectangular parallelopipeds, if their altitudes are each 6 inches, and their bases 5 inches by 4 inches, and 10 inches by 8 inches, respectively.

Ex. 638. Find the ratio of two rectangular parallelopipeds, if their dimensions are 3, 4, 5, and 9, 8, 10, respectively.

PROPOSITION X. THEOREM.

622. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions.*



Let a, b, and c be the three dimensions of the rectangular parallelopiped P, and let the cube U be the unit of volume.

To prove that the volume of $P = a \times b \times c$.

Proof. $$\frac{P}{U} = \frac{a \times b \times c}{1 \times 1 \times 1} = a \times b \times c. \qquad \S\ 621$$

Since U is the unit of volume, $\frac{P}{U}$ is the volume of P. § 601

Therefore, the volume of $P = a \times b \times c$. Q.E.D.

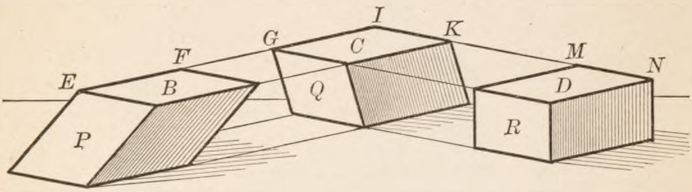
623. COR. 1. *The volume of a cube is the cube of its edge.*

624. COR. 2. *The volume of a rectangular parallelopiped is equal to the product of its base by its altitude.*

625. SCHOLIUM. When the three dimensions of a rectangular parallelopiped are each exactly divisible by the linear unit, this proposition is rendered evident by dividing the solid into cubes, each equal to the unit of volume. Thus, if the three edges which meet at a common vertex contain the linear unit 3, 5, and 8 times respectively, planes passed through the several points of division of the edges, perpendicular to the edges, will divide the solid into $3 \times 5 \times 8$ cubes, each equal to the unit of volume.

PROPOSITION XI. THEOREM.

626. *The volume of any parallelopiped is equal to the product of its base by its altitude.*



Let P be an oblique parallelopiped no two of whose faces are perpendicular, whose base B is a rhomboid, and whose altitude is H.

To prove that the volume of $P = B \times H$.

Proof. Prolong the edge EF and the edges ∥ to EF, and cut them perpendicularly by two parallel planes whose distance apart GI is equal to EF. We then have the oblique parallelopiped Q whose base C is a rectangle.

Prolong the edge IK and the edges ∥ to IK, and cut them perpendicularly by two planes whose distance apart MN is equal to IK. We then have the rectangular parallelopiped R.

Now $P \approxeq Q$, and $Q \approxeq R$. § 612

$\therefore P \approxeq R$. Ax. 1

The three solids have a common altitude H. § 530

Also $B \approxeq C$; § 401

and $C = D$. § 186

$\therefore B \approxeq D$. Ax. 1

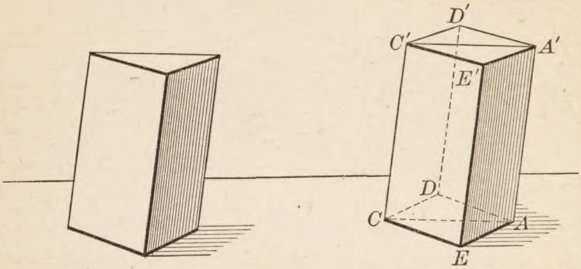
But the volume of $R = D \times H$. § 624

Putting P for R, and B for D, we have

the volume of $P = B \times H$. Q. E. D.

PROPOSITION XII. THEOREM.

627. *The volume of a triangular prism is equal to the product of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude of the triangular prism CEA-E'.

To prove that $V = B \times H.$

Proof. Upon the edges CE, EA, EE', construct the parallelopiped $CEAD\text{-}E'$.

Then $CEA\text{-}E' \approx \frac{1}{2}\, CEAD\text{-}E'.$ § 615

Now the volume of $CEAD\text{-}E' = CEAD \times H.$ § 626

But $CEAD = 2\,B.$ § 179

$\therefore V = \frac{1}{2}(2\,B \times H) = B \times H.$ Q.E.D.

Ex. 639. Two triangular prisms are equal if their lateral faces are equal, each to each, and similarly placed.

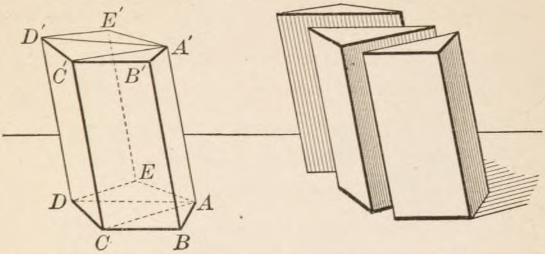
Ex. 640. The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of the three dimensions.

Ex. 641. The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of the twelve edges.

Ex. 642. The volume of a triangular prism is equal to half the product of any lateral face by the perpendicular dropped from the opposite edge on that face.

PROPOSITION XIII. THEOREM.

628. *The volume of any prism is equal to the product of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude of the prism DA′.

To prove that $V = B \times H.$

Proof. Planes passed through the lateral edge AA', and the diagonals AC, AD of the base, divide the given prism into triangular prisms that have the common altitude H.

The volume of each triangular prism is equal to the product of its base by its altitude (§ 627); and hence the sum of the volumes of the triangular prisms is equal to the sum of their bases multiplied by their common altitude.

But the sum of the triangular prisms is equal to the given prism, and the sum of their bases is equal to its base. Ax. 9

Therefore, the volume of the given prism is equal to the product of its base by its altitude.

That is, $V = B \times H.$ Q.E.D.

629. COR. *Two prisms are to each other as the products of their bases by their altitudes; prisms having equivalent bases are to each other as their altitudes; prisms having equal altitudes are to each other as their bases; prisms having equivalent bases and equal altitudes are equivalent.*

PROBLEMS OF COMPUTATION.

Ex. 643. If the edge of a cube is 15 inches, find the area of the total surface of the cube.

Ex. 644. If the length of a rectangular parallelopiped is 10 inches, its width 8 inches, and its height 6 inches, find the area of its total surface.

Ex. 645. Find the volume of a right triangular prism, if its height is 14 inches, and the sides of the base are 6, 5, and 5 inches.

Ex. 646. The base of a right prism is a rhombus, one side of which is 10 inches, and the shorter diagonal is 12 inches. The height of the prism is 15 inches. Find the entire surface and the volume.

Ex. 647. Find the volume of a regular prism whose height is 10 feet, if each side of its triangular base is 10 inches.

Ex. 648. How many square feet of lead will be required to line a cistern, open at the top, which is 4 feet 6 inches long, 2 feet 8 inches wide, and contains 42 cubic feet?

Ex. 649. An open cistern 6 feet long and $4\frac{1}{2}$ feet wide holds 108 cubic feet of water. How many square feet of lead will it take to line the sides and bottom?

Ex. 650. An open cistern is made of iron 2 inches thick. The inner dimensions are: length, 4 feet 6 inches; breadth, 3 feet; depth, 2 feet 6 inches. What will the cistern weigh (i) when empty? (ii) when full of water? (The specific gravity of the iron is 7.2.) → 2109.375 lbs

2109,375 + 4100,75 = 6210,125

Ex. 651. Find the volume of a regular hexagonal prism, if its height is 10 feet, and each side of the hexagon is 10 inches.

Ex. 652. Find the length of an edge of a cubical vessel that will hold 2 tons of water. $x = \sqrt[3]{\frac{4400}{62,5}} =$

Ex. 653. One edge of a cube is a. Find the surface, the volume, and the length of a diagonal of the cube.

Ex. 654. A diagonal of one of the faces of a cube is a. Find the volume of the cube.

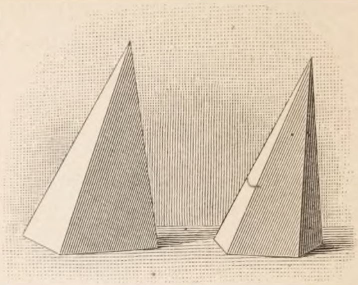
Ex. 655. The three dimensions of a rectangular parallelopiped are a, b, c. Find the area of its surface, its volume, and the length of a diagonal.

Ex. 656. The volume of a parallelopiped is V, and the three dimensions are as $m : n : p$. Find the dimensions.

PYRAMIDS.

630. DEF. A **pyramid** is a polyhedron of which one face, called the **base**, is a polygon of any number of sides and the other faces are triangles having a common vertex.

The faces which have a common vertex are called the **lateral faces** of the pyramid, and their common vertex is called the **vertex** of the pyramid.



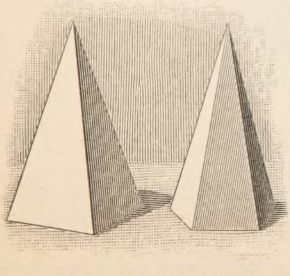
Pyramids.

631. DEF. The intersections of the lateral faces are called the **lateral edges** of the pyramid.

632. DEF. The sum of the areas of the lateral faces is called the **lateral area** of the pyramid.

633. DEF. The **altitude** of a pyramid is the length of the perpendicular let fall from the vertex to the plane of the base.

634. DEF. A pyramid is called **triangular, quadrangular,** etc., according as its base is a triangle, quadrilateral, etc.



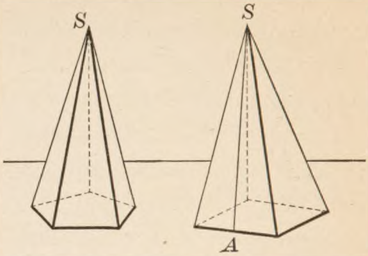
Regular Pyramids.

635. DEF. A triangular pyramid has four triangular faces, and is called a **tetrahedron**. Any one of its faces can be taken for its base.

636. DEF. A pyramid is **regular** if its base is a regular polygon whose centre coincides with the foot of the perpendicular let fall from the vertex to the base.

637. Def. The perpendicular let fall from the vertex to the base of a regular pyramid is called the **axis** of the pyramid.

The lateral edges of a regular pyramid are equal, for they cut off equal distances from the foot of the perpendicular let fall from the vertex to the base. § 514



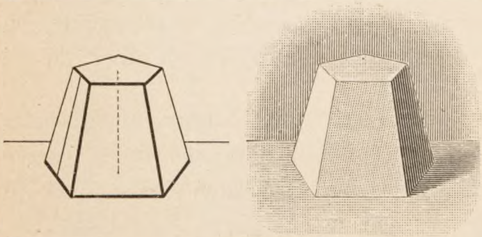
Therefore, the lateral faces of a regular pyramid are equal isosceles triangles. § 150

The altitudes of the lateral faces of a regular pyramid are equal.

638. Def. The **slant height of a regular pyramid** is the altitude of any one of the lateral faces. It bisects the base of the lateral face in which it is drawn. § 149

639. Def. A **truncated pyramid** is the portion of a pyramid included between the base and a section made by a plane cutting all the lateral edges.

A **frustum of a pyramid** is the portion of a pyramid included between the base and a section parallel to the base.



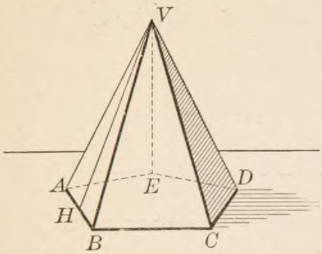
640. Def. The **altitude of a frustum** is the length of the perpendicular between the planes of its bases.

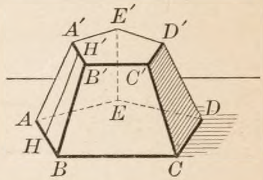
641. Def. The **lateral faces of a frustum** of a regular pyramid are equal isosceles trapezoids; and the sum of their areas is called the **lateral area of the frustum.**

642. Def. The **slant height of the frustum** of a regular pyramid is the altitude of one of these trapezoids.

PROPOSITION XIV. THEOREM.

643. *The lateral area of a regular pyramid is equal to half the product of its slant height by the perimeter of its base.*





Let S denote the lateral area of the regular pyramid V-ABCDE, L its slant height, and P the perimeter of its base.

To prove that $S = \frac{1}{2} L \times P$.

Proof. The $\triangle VAB$, VBC, etc., are equal isosceles $\triangle$. § 637

The area of each $\triangle$ is $\frac{1}{2} L$ multiplied by its base. § 403

$\therefore$ the sum of the areas of these $\triangle$ is $\frac{1}{2} L \times P$.

But the sum of the areas of these $\triangle$ is equal to S, the lateral area of the pyramid.

$$\therefore S = \frac{1}{2} L \times P. \qquad \text{Q.E.D.}$$

644. COR. *The lateral area of the frustum of a regular pyramid is equal to half the sum of the perimeters of the bases multiplied by the slant height of the frustum.* § 407

Ex. 657. Find the lateral area of a regular pyramid if the slant height is 16 feet, and the base is a regular hexagon with side 12 feet.

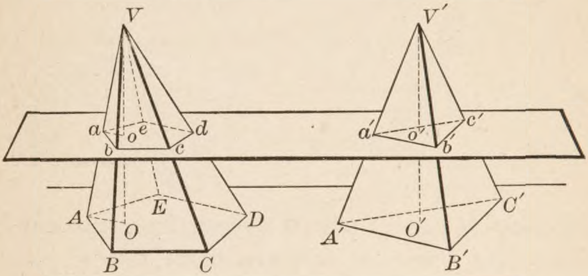
Ex. 658. Find the lateral area of a regular pyramid if the slant height is 8 feet, and the base is a regular pentagon with side 5 feet.

Ex. 659. Find the total surface of a regular pyramid if the slant height is 6 feet, and the base is a square with side 4 feet.

PROPOSITION XV. THEOREM.

645. *If a pyramid is cut by a plane parallel to the base:*

1. *The edges and altitude are divided proportionally.*
2. *The section is a polygon similar to the base.*



Let V-ABCDE be cut by a plane parallel to its base, intersecting the lateral edges in a, b, c, d, e, and the altitude in o.

1. *To prove that* $\frac{Va}{VA} = \frac{Vb}{VB} \cdots = \frac{Vo}{VO}$.

Proof. Since *abcde* is ∥ to *ABCDE*,

ab is ∥ to *AB*, *bc* to *BC* ⋯, and *ao* is ∥ to *AO*. § 528

$$\therefore \frac{Va}{VA} = \frac{Vb}{VB} \cdots = \frac{Vo}{VO}. \qquad \S\ 343$$

2. *To prove the section abcde similar to the base ABCDE.*

Proof. Since △ *Vab* is similar to △ *VAB*,

△ *Vbc* is similar to △ *VBC*, etc., § 354

$$\frac{ab}{AB} = \left(\frac{Vb}{VB}\right) = \frac{bc}{BC} = \left(\frac{Vc}{VC}\right) = \frac{cd}{CD}, \text{ etc.} \qquad \S\ 351$$

∴ the homologous sides of the polygons are proportional.

Since ab is $\parallel$ to AB, bc to BC, cd to CD, etc., § 528

$\angle abc = \angle ABC$, $\angle bcd = \angle BCD$, etc. § 534

Therefore, the two polygons are mutually equiangular.

Hence, section $abcde$ is similar to the base $ABCDE$. § 351

Q. E. D.

646. Cor. 1. *Any section of a pyramid parallel to the base is to the base as the square of the distance from the vertex is to the square of the altitude of the pyramid.*

For $$\frac{Vo}{VO} = \left(\frac{Va}{VA}\right) = \frac{ab}{AB}. \quad \therefore \frac{\overline{Vo}^2}{\overline{VO}^2} = \frac{\overline{ab}^2}{\overline{AB}^2}.$$ § 338

But $$\frac{abcde}{ABCDE} = \frac{\overline{ab}^2}{\overline{AB}^2}.$$ § 412

$$\therefore \frac{abcde}{ABCDE} = \frac{\overline{Vo}^2}{\overline{VO}^2}.$$ Ax. 1

647. Cor. 2. *If two pyramids having equal altitudes are cut by planes parallel to the bases, and at equal distances from the vertices, the sections have the same ratio as the bases.*

For $$\frac{abcde}{ABCDE} = \frac{\overline{Vo}^2}{\overline{VO}^2},$$

and $$\frac{a'b'c'}{A'B'C'} = \frac{\overline{V'o'}^2}{\overline{V'O'}^2}.$$ § 646

But $Vo = V'o'$, and $VO = V'O'$. Hyp.

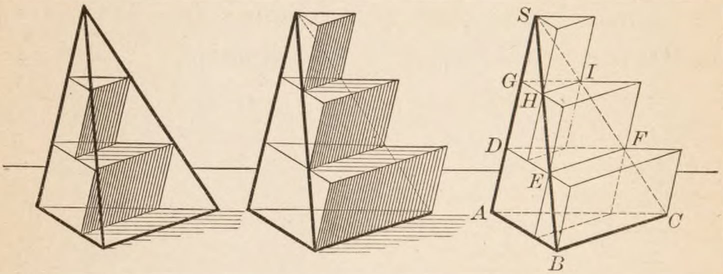
$\therefore abcde : ABCDE = a'b'c' : A'B'C'$.

Whence $abcde : a'b'c' = ABCDE : A'B'C'$. § 330

648. Cor. 3. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to the bases, and at equal distances from the vertices, are equivalent.*

PROPOSITION XVI. THEOREM.

649. *The volume of a triangular pyramid is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms of equal altitude, if the number of prisms is indefinitely increased.*



Let S-ABC be a triangular pyramid.

To prove that its volume is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms of equal altitude, if the number of prisms is indefinitely increased.

Proof. Divide the altitude into n equal parts, and denote the length of each part by h.

Through the points of division pass planes parallel to the base, cutting the pyramid in the sections DEF, GHI, etc.

Through EF, HI, etc., pass planes parallel to SA, thus forming a series of prisms which have DEF, GHI, etc., for *upper* bases, and h for altitude.

These prisms may be said to be *inscribed in* the pyramid.

Again, through BC, EF, etc., pass planes parallel to SA, thus forming a series of prisms which have ABC, DEF, etc., for *lower* bases, and h for altitude.

These prisms may be said to be *circumscribed about* the pyramid.

Each inscribed prism is equal to the circumscribed prism directly above it. § 629

The difference, therefore, between the sum of the volumes of the inscribed prisms and the sum of the volumes of the circumscribed prisms is the prism $D\text{-}ABC$.

Denote the volume of the pyramid by P,

the sum of the volumes of the inscribed prisms by V,

the sum of the volumes of the circumscribed prisms by V',

and the difference between these two sums by d.

Then $$V' - V = d.$$

By increasing n indefinitely, and consequently decreasing h indefinitely, d can be made less than any assigned quantity, however small, but cannot be made zero.

Therefore, $V' - V$ can be made less than any assigned quantity, however small, but cannot be made zero.

Now $$V' > P, \text{ and } V < P. \qquad \text{Ax. 8}$$

Therefore, the difference between P and either V' or V is less than $V' - V$.

Therefore, $V' - P$ can be made less than any assigned quantity, however small, but cannot be made zero.

And $P - V$ can be made less than any assigned quantity, however small, but cannot be made zero.

Therefore, P is the common limit of V' and V. § 275

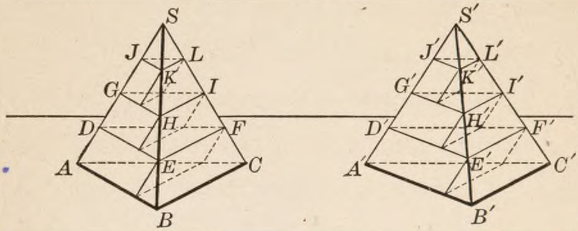
Q. E. D.

Ex. 660. The slant height of a regular pyramid is divided in the ratio 1 : 3 by a plane parallel to the base. Find the ratio of the base to the section.

Ex. 661. The section of a pyramid made by a plane parallel to the base is half as large as the base. Find the ratio of the segments into which the altitude is divided by the plane.

PROPOSITION XVII. THEOREM.

650. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*



Let S-ABC and S′-A′B′C′ be two triangular pyramids having equivalent bases situated in the same plane, and equal altitudes.

To prove that $S\text{-}ABC \approx S'\text{-}A'B'C'$.

Proof. Divide the altitude into n equal parts, and through the points of division pass planes parallel to the plane of the bases, forming the sections DEF, GHI, etc., $D'E'F'$, $G'H'I'$, etc.

In the pyramids $S\text{-}ABC$ and $S'\text{-}A'B'C'$ inscribe prisms whose upper bases are the sections DEF, GHI, etc., $D'E'F'$, $G'H'I'$, etc.

The corresponding sections are equivalent. § 648

Therefore, the corresponding prisms are equivalent. § 629

Denote the sum of the volumes of the prisms inscribed in the pyramid $S\text{-}ABC$ by V, and the sum of the volumes of the corresponding prisms inscribed in the pyramid $S'\text{-}A'B'C'$ by V'.

Then $V = V'$. Ax. 2

Now let the number of equal parts into which the altitude is divided be indefinitely increased.

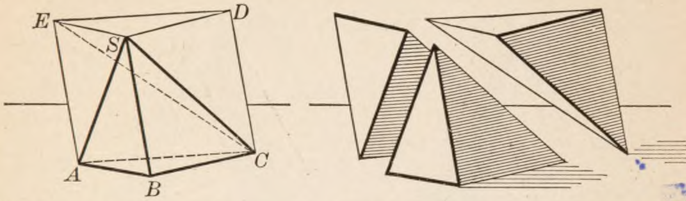
The volumes V and V' are always equal, and approach as limits the pyramids $S\text{-}ABC$ and $S'\text{-}A'B'C'$, respectively. § 649

Hence, $S\text{-}ABC \approx S'\text{-}A'B'C'$. § 284

Q.E.D.

Proposition XVIII. Theorem.

651. *The volume of a triangular pyramid is equal to one third of the product of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude, of the triangular pyramid S-ABC.

To prove that $V = \frac{1}{3} B \times H$.

Proof. On the base ABC construct a prism $ABC\text{-}ESD$, having its lateral edges equal and parallel to SB.

The prism is composed of the triangular pyramid $S\text{-}ABC$ and the quadrangular pyramid $S\text{-}ACDE$.

Through SE and SC pass a plane SEC.

This plane divides the quadrangular pyramid into the two triangular pyramids $S\text{-}ACE$ and $S\text{-}DEC$, which have the same altitude and equal bases. § 179

$\therefore S\text{-}ACE \approx S\text{-}DEC$. § 650

The pyramid $S\text{-}DEC$ may be regarded as having ESD for its base and C for its vertex, and is, therefore, equivalent to $S\text{-}ABC$. § 650

Hence, the three pyramids into which the prism $ABC\text{-}ESD$ is divided are equivalent. Ax. 1

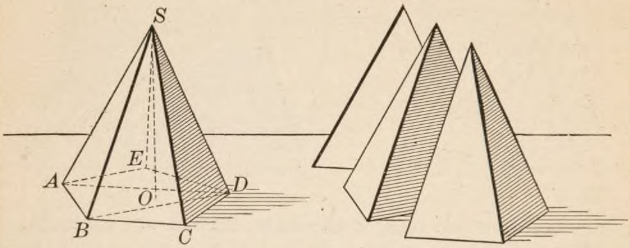
$\therefore$ the pyramid $S\text{-}ABC$ is equivalent to one third of the prism.

But the volume of the prism is equal to $B \times H$. § 627

$\therefore V = \frac{1}{3} B \times H$. Q.E.D

PROPOSITION XIX. THEOREM.

652. *The volume of any pyramid is equal to one third the product of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude of the pyramid S-ABCDE.

To prove that $V = \frac{1}{3} B \times H$.

Proof. Through the edge SD, and the diagonals of the base DA, DB, pass planes.

These divide the pyramid into triangular pyramids, whose bases are the triangles which compose the base of the pyramid, and whose volumes are together equal to one third the sum of their bases multiplied by their common altitude, H. § 651

That is, $V = \frac{1}{3} B \times H$. Q.E.D.

653. COR. 1. *The volumes of two pyramids are to each other as the products of their bases and altitudes; pyramids of equivalent bases are to each other as their altitudes, and of equal altitudes are to each other as their bases; pyramids having equivalent bases and equal altitudes are equivalent.*

654. COR. 2. *The volume of any polyhedron may be found by dividing it into pyramids, computing their volumes separately, and finding the sum of their volumes.*

PROBLEMS OF COMPUTATION.

Find the volume in cubic feet of a regular pyramid:

Ex. 662. When its base is a square, each side measuring 3 feet 4 inches, and its height is 9 feet.

Ex. 663. When its base is an equilateral triangle, each side measuring 10 feet, and its height is 15 feet.

Ex. 664. When its base is a regular hexagon, each side measuring 4 feet, and its height is 20 feet.

Find the total surface in square feet of a regular pyramid:

Ex. 665. When each side of its square base is 10 feet, and the slant height is 18 feet.

Ex. 666. When each side of its triangular base is 8 feet, and the slant height is 16 feet.

Ex. 667. When each side of its square base is 32 feet, and the perpendicular height is 72 feet.

Find the height in feet of a pyramid when:

Ex. 668. The volume is 26 cubic feet 936 cubic inches, and each side of its square base is 3 feet 6 inches.

Ex. 669. The volume is 20 cubic feet, and the sides of its triangular base are 5 feet, 4 feet, and 3 feet.

Ex. 670. The base edge of a regular pyramid with a square base measures 40 feet, the lateral edge 101 feet. Find its volume in cubic feet.

Ex. 671. Find the volume of a regular pyramid whose slant height is 12 feet, and whose base is an equilateral triangle inscribed in a circle that has a radius of 10 feet.

Ex. 672. Having given the base edge a, and the total surface T, of a regular pyramid with a square base, find the volume V.

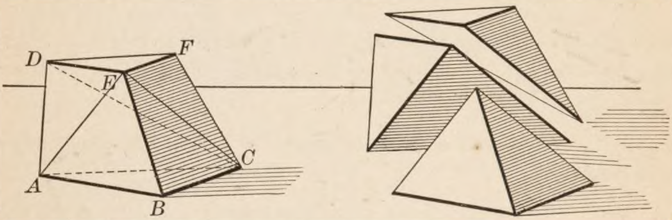
Ex. 673. The base edge of a regular pyramid whose base is a square is a, the total surface T. Find the height of the pyramid.

Ex. 674. The eight edges of a regular pyramid with a square base are equal in length, and the total surface is T. Find the length of one edge.

Ex. 675. Find the base edge a of a regular pyramid with a square base, having given the height H and the total surface T.

PROPOSITION XX. THEOREM.

655. *The frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and the mean proportional between the two bases of the frustum.*



Let B and b denote the lower and upper bases of the frustum ABC-DEF, and H its altitude.

Through the vertices A, E, C and E, D, C pass planes dividing the frustum into three pyramids.

Now the pyramid $E\text{-}ABC$ has for its altitude H, the altitude of the frustum, and for its base B, the lower base of the frustum.

And the pyramid $C\text{-}EDF$ has for its altitude H, the altitude of the frustum, and for its base b, the upper base of the frustum. Hence, it remains

To prove $E\text{-}ADC$ equivalent to a pyramid, having for its altitude H, and for its base $\sqrt{B \times b}$.

Proof. $E\text{-}ABC$ and $E\text{-}ADC$, regarded as having the common vertex C, and their bases in the same plane BD, have a common altitude.

$$\therefore C\text{-}ABE : C\text{-}ADE = \triangle AEB : \triangle AED. \qquad \S\ 653$$

Now the $\triangle$ AEB and AED have a common altitude, the altitude of the trapezoid $ABED$.

$$\therefore \triangle AEB : \triangle AED = AB : DE. \qquad \text{§ 405}$$

$$\therefore C\text{-}ABE : C\text{-}ADE = AB : DE. \qquad \text{Ax. 1}$$

That is, $E\text{-}ABC : E\text{-}ADC = AB : DE.$

In like manner $E\text{-}ADC$ and $E\text{-}DFC$, regarded as having the common vertex E, and their bases in the same plane DC, have a common altitude.

$$\therefore E\text{-}ADC : E\text{-}DFC = \triangle ADC : \triangle DFC. \qquad \text{§ 653}$$

But since the $\triangle$ ADC and DFC have a common altitude, the altitude of the trapezoid $ACFD$,

$$\therefore \triangle ADC : \triangle DFC = AC : DF. \qquad \text{§ 405}$$

$$\therefore E\text{-}ADC : E\text{-}DFC = AC : DF. \qquad \text{Ax. 1}$$

But $\triangle DEF$ is similar to $\triangle ABC$. § 645

$$\therefore AB : DE = AC : DF. \qquad \text{§ 351}$$

$$\therefore E\text{-}ABC : E\text{-}ADC = E\text{-}ADC : E\text{-}DFC. \qquad \text{Ax. 1}$$

Now $E\text{-}ABC = \frac{1}{3} H \times B,$ § 651

and $E\text{-}DFC = C\text{-}EDF = \frac{1}{3} H \times b.$

$$\therefore E\text{-}ADC = \sqrt{\tfrac{1}{3} H \times B \times \tfrac{1}{3} H \times b} = \tfrac{1}{3} H \sqrt{B \times b}.$$

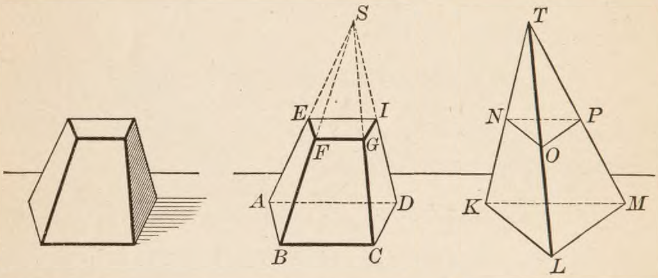
Hence, $E\text{-}ADC$ is equivalent to a pyramid, having for its altitude H, and for its base $\sqrt{B \times b}$. Q.E.D.

656. Cor. *If the volume of a frustum of a triangular pyramid is denoted by* V, *the lower base by* B, *the upper base by* b, *and the altitude by* H,

$$V = \tfrac{1}{3} H \times B + \tfrac{1}{3} H \times b + \tfrac{1}{3} H \times \sqrt{B \times b}$$
$$= \tfrac{1}{3} H \times (B + b + \sqrt{B \times b}).$$

Proposition XXI. Theorem.

657. *The volume of the frustum of any pyramid is equal to the sum of the volumes of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and the mean proportional between the bases of the frustum.*



Let B and b denote the lower and upper bases, H the altitude, and V the volume of the frustum ABCD-EFGI.

To prove that $V = \frac{1}{3}H(B + b + \sqrt{B \times b})$.

Proof. Let $T\text{-}KLM$ be a triangular pyramid having the same altitude as $S\text{-}ABCD$ and its base KLM equivalent to $ABCD$, and lying in the same plane. Then $T\text{-}KLM \approx S\text{-}ABCD$. § 653

Let the plane $EFGI$ cut $T\text{-}KLM$ in NOP.

Then $NOP \approx EFGI$. § 648

Hence, $T\text{-}NOP \approx S\text{-}EFGI$. § 650

Hence, if we take away the upper pyramids, we have left the equivalent frustums $NOP\text{-}KLM$ and $EFGI\text{-}ABCD$.

But the volume of the frustum $NOP\text{-}KLM$ is equal to

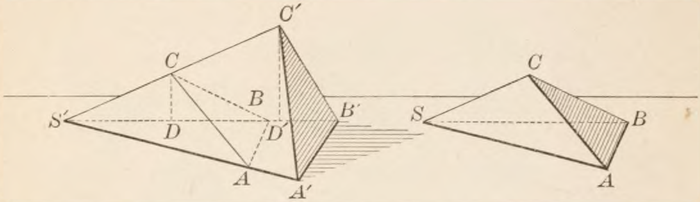
$$\tfrac{1}{3}H(B + b + \sqrt{B \times b}). \qquad \text{§ 656}$$

$$\therefore V = \tfrac{1}{3}H(B + b + \sqrt{B \times b}).$$

Q. E. D.

PROPOSITION XXII. THEOREM.

658. *The volumes of two triangular pyramids, having a trihedral angle of the one equal to a trihedral angle of the other, are to each other as the products of the three edges of these trihedral angles.*



Let V and V′ denote the volumes of the two triangular pyramids S-ABC and S′-A′B′C′, having the trihedral angles S and S′ equal.

To prove that $\dfrac{V}{V'} = \dfrac{SA \times SB \times SC}{S'A' \times S'B' \times S'C'}$.

Proof. Place the pyramid $S\text{-}ABC$ upon $S'\text{-}A'B'C'$ so that the trihedral $\angle S$ shall coincide with S'.

Draw CD and $C'D' \perp$ to the plane $S'A'B'$,

and let their plane intersect $S'A'B'$ in $S'DD'$.

The faces $S'AB$ and $S'A'B'$ may be taken as the bases, and CD, $C'D'$ as the altitudes, of the triangular pyramids $C\text{-}S'AB$ and $C'\text{-}S'A'B'$, respectively.

Then $$\frac{V}{V'} = \frac{S'AB \times CD}{S'A'B' \times C'D'} = \frac{S'AB}{S'A'B'} \times \frac{CD}{C'D'}. \qquad \S\ 653$$

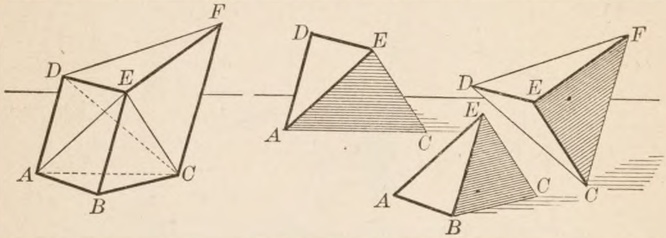
But $$\frac{S'AB}{S'A'B'} = \frac{S'A \times S'B}{S'A' \times S'B'}, \qquad \S\ 410$$

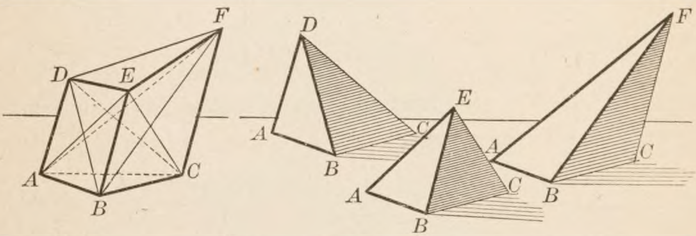
and $$\frac{CD}{C'D'} = \frac{S'C}{S'C'}. \qquad \S\ 351$$

$$\therefore \frac{V}{V'} = \frac{S'A \times S'B \times S'C}{S'A' \times S'B' \times S'C'} = \frac{SA \times SB \times SC}{S'A' \times S'B' \times S'C'}.$$ Q. E. D.

Proposition XXIII. Theorem.

659. *A truncated triangular prism is equivalent to the sum of three pyramids, whose common base is the base of the prism and whose vertices are the three vertices of the inclined section.*





Let ABC-DEF be a truncated triangular prism whose base is ABC, and inclined section DEF.

Pass the planes AEC and DEC, dividing the truncated prism into the three pyramids $E\text{-}ABC$, $E\text{-}ACD$, and $E\text{-}CDF$.

To prove ABC-DEF equivalent to the sum of the three pyramids, E-ABC, D-ABC, and F-ABC.

Proof. $E\text{-}ABC$ has the base ABC and the vertex E.

The pyramid $E\text{-}ACD \approx B\text{-}ACD$. § 650

For they have the same base ACD, and the same altitude since their vertices E and B are in the line $EB \parallel$ to ACD.

But the pyramid B-ACD may be regarded as having the base ABC and the vertex D; that is, as D-ABC.

The pyramid E-$CDF \approx B$-ACF. § 650

For their bases CDF and ACF are equivalent, § 404

since the $\triangle CDF$ and ACF have the common base CF and equal altitudes, their vertices lying in the line $AD \parallel$ to CF; and the pyramids have the same altitude, since their vertices E and B are in the line $EB \parallel$ to the plane of their bases.

But the pyramid B-ACF may be regarded as having the base ABC and the vertex F; that is, as F-ABC.

Therefore, the truncated triangular prism ABC-DEF is equivalent to the sum of the three pyramids E-ABC, D-ABC, and F-ABC. Q.E.D.

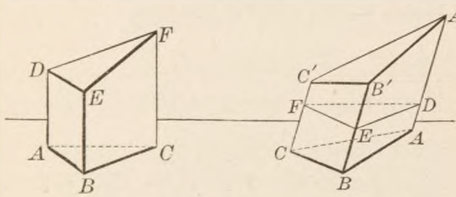


FIG. 1. FIG. 2.

660. COR. 1. *The volume of a truncated right triangular prism is equal to the product of its base by one third the sum of its lateral edges.*

For the lateral edges DA, EB, FC (Fig. 1), being perpendicular to the base ABC, are the altitudes of the three pyramids whose sum is equivalent to the truncated prism.

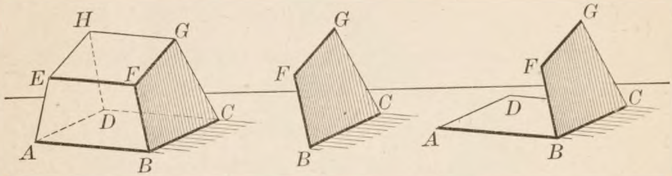
661. COR. 2. *The volume of any truncated triangular prism is equal to the product of its right section by one third the sum of its lateral edges.*

For the right section DEF (Fig. 2) divides the truncated prism into two truncated right prisms.

GENERAL THEOREMS OF POLYHEDRONS.

Proposition XXIV. Theorem. (Euler's.)

662. *In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.*



Let E denote the number of edges, V the number of vertices, F the number of faces, of the polyhedron AG.

To prove that $E + 2 = V + F.$

Proof. Beginning with one face $BCGF$, we have $E = V$.

Annex a second face $ABCD$, by applying one of its edges to a corresponding edge of the first face, and there is formed a surface of two faces, having *one* edge BC and *two* vertices B and C common to the two faces.

Therefore, for 2 faces $E = V + 1$.

Annex a third face $ABFE$, adjoining each of the first two faces; this face will have *two* edges, AB, BF, and *three* vertices, A, B, F, in common with the surface already formed.

Therefore, for 3 faces $E = V + 2$.

In like manner, for 4 faces $E = V + 3$, and so on.

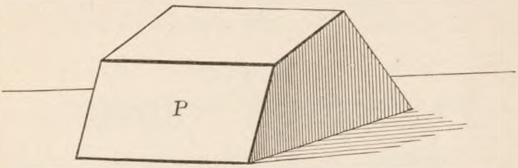
Therefore, for $(F - 1)$ faces $E = V + (F - 2)$.

But $F - 1$ is the number of faces of the polyhedron when only one face is lacking, and the addition of this face will not increase the number of edges or vertices. Hence, for F faces

$$E = V + F - 2, \text{ or } E + 2 = V + F. \qquad \text{Q.E.D.}$$

Proposition XXV. Theorem.

663. *The sum of the face angles of any polyhedron is equal to four right angles taken as many times, less two, as the polyhedron has vertices.*



Let E denote the number of edges, V the number of vertices, F the number of faces, and S the sum of the face angles, of the polyhedron P.

To prove that $S = (V - 2)\,4\ rt.\ \angle s.$

Proof. Since E denotes the number of edges, $2\,E$ will denote the number of sides of the faces, considered as independent polygons, for each edge is common to two polygons.

If an exterior angle is formed at each vertex of every polygon, the sum of the interior and exterior angles at each vertex is 2 rt. $\angle s$ (§ 86); and since there are $2\,E$ vertices, the sum of the interior and exterior angles of all the faces is

$$2\,E \times 2 \text{ rt. } \angle s, \text{ or } E \times 4 \text{ rt. } \angle s.$$

But the sum of the ext. $\angle s$ of each face is 4 rt. $\angle s$. § 207

Therefore, the sum of all the ext. $\angle s$ of F faces is

$$F \times 4 \text{ rt. } \angle s.$$

Therefore, $S = E \times 4 \text{ rt. } \angle s - F \times 4 \text{ rt. } \angle s$

$= (E - F)\,4 \text{ rt. } \angle s.$

But $E + 2 = V + F$; § 626

that is, $E - F = V - 2.$

Therefore, $S = (V - 2)\,4 \text{ rt. } \angle s.$ Q.E.D.

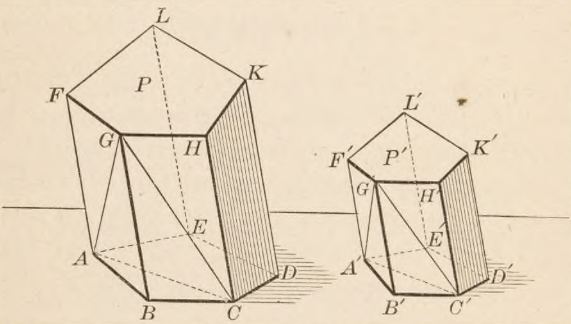
SIMILAR POLYHEDRONS.

664. Def. **Similar polyhedrons** are polyhedrons that have the same number of faces, respectively similar and similarly placed, and have their corresponding polyhedral angles equal.

665. Def. Homologous faces, lines, and angles of similar polyhedrons are faces, lines, and angles similarly situated.

Proposition XXVI. Theorem.

666. *Two similar polyhedrons may be decomposed into the same number of tetrahedrons similar, each to each, and similarly placed.*



Let P and P′ be two similar polyhedrons with G and G′ homologous vertices.

To prove that P and P′ can be decomposed into the same number of tetrahedrons, similar each to each, and similarly placed.

Proof. Divide all the faces of P and P', except those which include the angles G and G', into corresponding $\triangle$.

Pass planes through G and the vertices of the $\triangle$ in P; also through G' and the vertices of the $\triangle$ in P'.

Any two corresponding tetrahedrons $G\text{-}ABC$ and $G'\text{-}A'B'C'$ are similar; for they have the faces ABC, GAB, GBC, similar, respectively, to $A'B'C'$, $G'A'B'$, $G'B'C'$; § 365

and the face GAC similar to $G'A'C'$, § 358

since $$\frac{AG}{A'G'}=\left(\frac{AB}{A'B'}\right)=\frac{AC}{A'C'}=\left(\frac{BC}{B'C'}\right)=\frac{GC}{G'C'}.$$ § 351

They also have the corresponding trihedral $\angle$s equal. § 582

$\therefore$ the tetrahedron $G\text{-}ABC$ is similar to $G'\text{-}A'B'C'$. § 664

If $G\text{-}ABC$ and $G'\text{-}A'B'C'$ are removed, the polyhedrons remaining continue similar; for the new faces GAC and $G'A'C'$ have just been proved similar, and the modified faces AGF and $A'G'F'$, CGH and $C'G'H'$, are similar (§ 365); also the modified polyhedral $\angle$s G and G', A and A', C and C', remain equal each to each, since the corresponding parts taken from them are equal.

The process of removing similar tetrahedrons can be carried on until the polyhedrons are decomposed into the same number of tetrahedrons similar each to each, and similarly placed.

Q.E.D.

667. Cor. 1. *The homologous edges of similar polyhedrons are proportional.* § 351

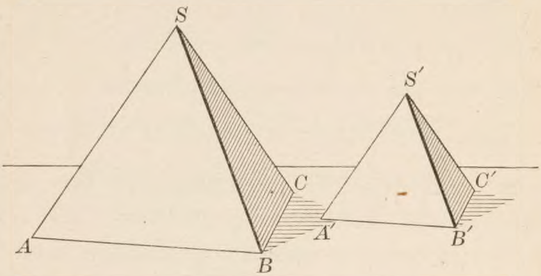
668. Cor. 2. *Any two homologous lines in two similar polyhedrons have the same ratio as any two homologous edges.* § 351

669. Cor. 3. *Two homologous faces of similar polyhedrons are proportional to the squares of two homologous edges.* § 412

670. Cor. 4. *The entire surfaces of two similar polyhedrons are proportional to the squares of two homologous edges.* § 335

PROPOSITION XXVII. THEOREM.

671. *The volumes of two similar tetrahedrons are to each other as the cubes of their homologous edges.*



Let V and V′ denote the volumes of the two similar tetrahedrons S-ABC and S′-A′B′C′.

To prove that $$\frac{V}{V'} = \frac{\overline{SB}^3}{\overline{S'B'}^3}.$$

Proof. $$\frac{V}{V'} = \frac{SB \times SC \times SA}{S'B' \times S'C' \times S'A'} \qquad \S\ 658$$

$$= \frac{SB}{S'B'} \times \frac{SC}{S'C'} \times \frac{SA}{S'A'}.$$

But $$\frac{SB}{S'B'} = \frac{SC}{S'C'} = \frac{SA}{S'A'}. \qquad \S\ 667$$

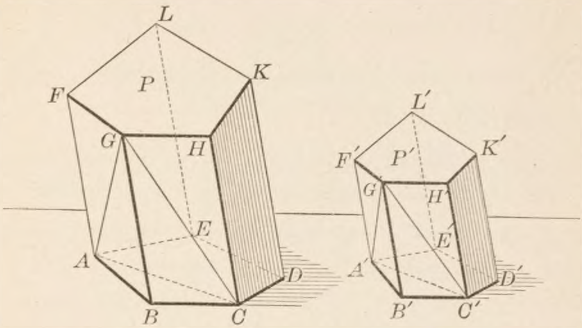
$$\therefore \frac{V}{V'} = \frac{SB}{S'B'} \times \frac{SB}{S'B'} \times \frac{SB}{S'B'} = \frac{\overline{SB}^3}{\overline{S'B'}^3}.$$ Q. E. D.

Ex. 676. The homologous edges of two similar tetrahedrons are as 6 : 7. Find the ratio of their surfaces and of their volumes.

Ex. 677. If the edge of a tetrahedron is a, find the homologous edge of a similar tetrahedron twice as large.

Proposition XXVIII. Theorem.

672. *The volumes of two similar polyhedrons are to each other as the cubes of any two homologous edges.*



Let V, V′ denote the volumes, GB, G′B′ any two homologous edges, of the polyhedrons P and P′.

To prove that $\quad V : V' = \overline{GB}^3 : \overline{G'B'}^3.$

Proof. Decompose these polyhedrons into tetrahedrons similar, each to each, and similarly placed. § 666

Denote the volumes of these tetrahedrons by $v, v_1, v_2, \cdots,$ $v', v_1', v_2', \cdots$

Then $\dfrac{v}{v'} = \dfrac{\overline{GB}^3}{\overline{G'B'}^3}, \quad \dfrac{v_1}{v_1'} = \dfrac{\overline{GB}^3}{\overline{G'B'}^3}, \quad \dfrac{v_2}{v_2'} = \dfrac{\overline{GB}^3}{\overline{G'B'}^3},$ and so on. § 671

$$\therefore \frac{v}{v'} = \frac{v_1}{v_1'} = \frac{v_2}{v_2'} = \cdots$$

Whence $\quad \dfrac{v + v_1 + v_2 + \cdots}{v' + v_1' + v_2' + \cdots} = \dfrac{v}{v'} = \dfrac{\overline{GB}^3}{\overline{G'B'}^3}.$ § 335

That is, $\quad \dfrac{V}{V'} = \dfrac{\overline{GB}^3}{\overline{G'B'}^3}.$

Q.E.D.

REGULAR POLYHEDRONS.

673. Def. A **regular polyhedron** is a polyhedron whose faces are equal regular polygons, and whose polyhedral angles are equal.

Proposition XXIX. Problem.

674. *To determine the number of regular convex polyhedrons possible.*

A convex polyhedral angle must have at least three faces, and the sum of its face angles must be less than 360° (§ 581).

1. Since each angle of an equilateral triangle is 60°, convex polyhedral angles may be formed by combining three, four, or five equilateral triangles. The sum of six such angles is 360°, and hence greater than the sum of the face angles of a convex polyhedral angle. Hence, three regular convex polyhedrons are possible with equilateral triangles for faces.

2. Since each angle of a square is 90°, a convex polyhedral angle may be formed by combining three squares. The sum of four such angles is 360°, and therefore greater than the sum of the face angles of a convex polyhedral angle. Hence, one regular convex polyhedron is possible with squares.

3. Since each angle of a regular pentagon is 108° (§ 206), a convex polyhedral angle may be formed by combining three regular pentagons. The sum of four such angles is 432°, and therefore greater than the sum of the face angles of a convex polyhedral angle. Hence, one regular convex polyhedron is possible with regular pentagons.

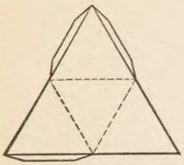
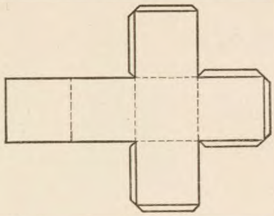
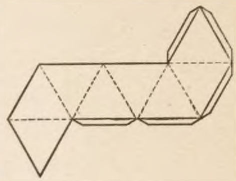
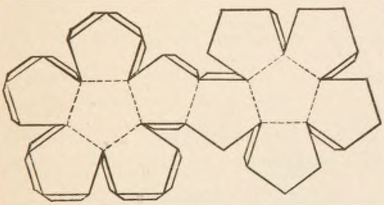
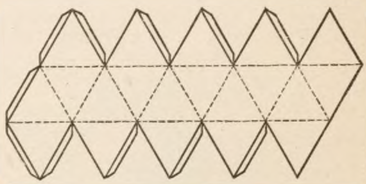
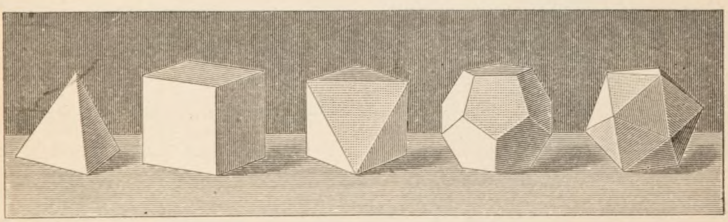
4. The sum of three angles of a regular hexagon is 360°, of a regular heptagon is greater than 360°, etc. Hence, only five regular convex polyhedrons are possible.

The five regular polyhedrons are called the **tetrahedron**, the **hexahedron**, the **octahedron**, the **dodecahedron**, the **icosahedron**.

Q. E. F.

675. The regular polyhedrons may be constructed as follows:

Draw the diagrams given below on stiff paper. Cut through the full lines and paste strips of paper on the edges shown in the diagrams. Fold on the dotted lines, and keep the edges in contact by the pasted strips of paper.

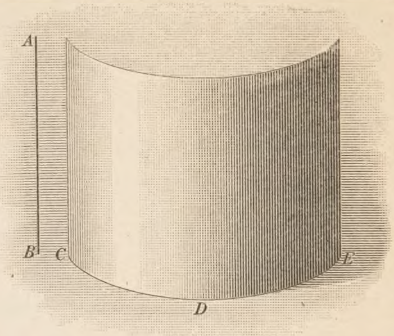







Tetrahedron. Hexahedron. Octahedron. Dodecahedron. Icosahedron.

CYLINDERS.

676. Def. A **cylindrical surface** is a curved surface generated by a straight line, which moves parallel to a fixed straight line and constantly touches a fixed curve not in the plane of the straight line.

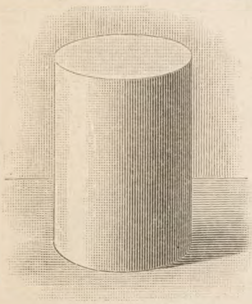
The moving line is called the **generatrix**, and the fixed curve the **directrix**.



Cylindrical Surface.

677. Def. The generatrix in any position is called an **element of the cylindrical surface**.

678. Def. A **cylinder** is a solid bounded by a cylindrical surface and two parallel plane surfaces.


Right Cylinder.

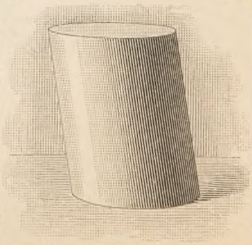
679. Def. The two plane surfaces are called the **bases**, and the cylindrical surface is called the **lateral surface**.

680. Def. The **altitude of a cylinder** is the perpendicular distance between the planes of its bases. The elements of a cylinder are all equal.

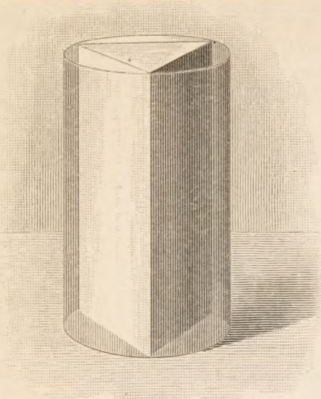
681. Def. A cylinder is a **right cylinder** if its elements are perpendicular to its bases; otherwise, an **oblique cylinder**.

682. Def. A **circular cylinder** is a cylinder whose bases are circles.

683. Def. A right circular cylinder is called a **cylinder of revolution** because


Oblique Cylinder.

it may be generated by the revolution of a rectangle about one side as an **axis.**



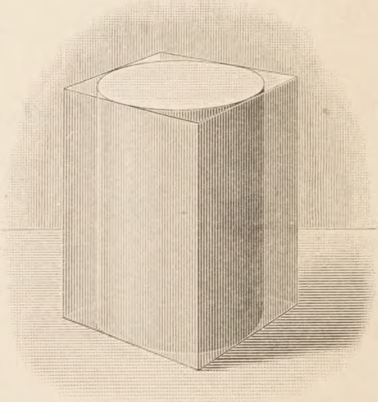
Inscribed Prism.

684. Def. **Similar cylinders of revolution** are cylinders generated by the revolution of similar rectangles about homologous sides.

685. Def. A **tangent line to a cylinder** is a straight line, not an element, which touches the lateral surface of the cylinder but does not intersect it.

686. Def. A **tangent plane to a cylinder** is a plane which contains an element of the cylinder but does not cut the surface.

The element contained by the plane is called the **element of contact.**



Circumscribed Prism.

687. Def. A **prism is inscribed in a cylinder** when its lateral edges are elements of the cylinder and its bases are inscribed in the bases of the cylinder.

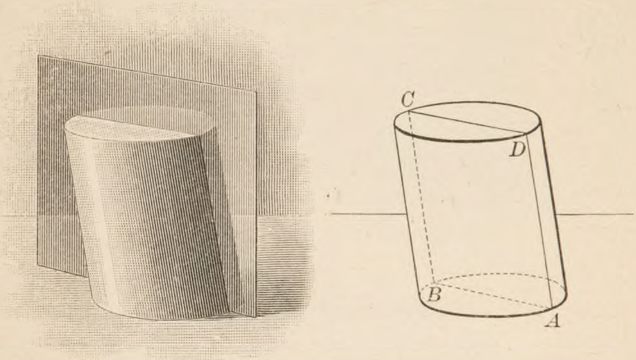
688. Def. A **prism is circumscribed about a cylinder** when its lateral edges are parallel to the elements of the cylinder and its bases are circumscribed about the bases of the cylinder.

689. Def. A **section of a cylinder** is the figure formed by its intersection with a plane passing through it.

A **right section of a cylinder** is a section made by a plane perpendicular to its elements.

PROPOSITION XXX. THEOREM.

690. *Every section of a cylinder made by a plane passing through an element is a parallelogram.*



Let ABCD be a section of the cylinder AC made by a plane passing through the element AD.

To prove that ABCD is a parallelogram.

Proof. Through B draw a line in the plane AC, $\parallel$ to AD.

This line is an element of the cylindrical surface. § 676

Since this line is in both the plane and the cylindrical surface, it must be their intersection and coincide with BC.

Hence, BC coincides with a straight line parallel to AD.

Therefore, BC is a straight line $\parallel$ to AD.

Also AB is a straight line $\parallel$ to CD. § 528

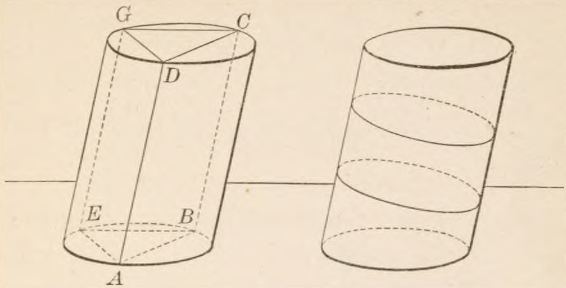
$\therefore$ $ABCD$ is a parallelogram. § 166

Q.E.D.

691. COR. *Every section of a right cylinder made by a plane passing through an element is a rectangle.*

PROPOSITION XXXI. THEOREM.

692. *The bases of a cylinder are equal.*



Let ABE and DCG be the bases of the cylinder AC.

To prove that $ABE = DCG$.

Proof. Let A, B, E be any three points in the perimeter of the lower base, and AD, BC, EG be elements of the surface.

Draw AE, AB, EB, DG, DC, GC.

Then AC, AG, EC are ▱s. § 183

$\therefore AE = DG$, $AB = DC$, and $EB = GC$. § 178

$\therefore \triangle ABE = \triangle DCG$. § 150

Place the lower base on the upper base so that the $\triangle ABE$ shall fall on the $\triangle DCG$. Then A, B, E will fall on D, C, G. But A, B, E are *any* points in the perimeter of the lower base. Therefore, *all* points in the perimeter of the lower base will fall on the perimeter of the upper base, and the bases will coincide and be equal. Q. E. D.

693. COR. 1. *Any two parallel sections, cutting all the elements of a cylinder, are equal.*

For these sections are the bases of the included cylinder.

694. COR. 2. *Any section of a cylinder parallel to the base is equal to the base.*

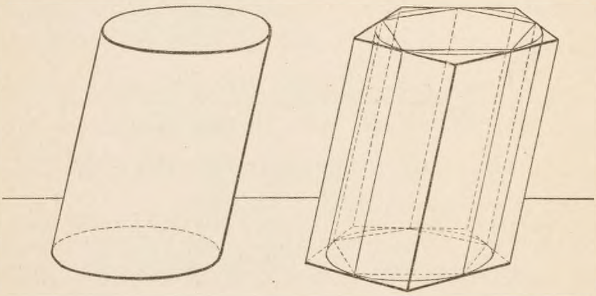
PROPOSITION XXXII. THEOREM.

695. *If a prism whose base is a regular polygon is inscribed in or circumscribed about a circular cylinder, and if the number of sides of the base of the prism is indefinitely increased,*

The volume of the cylinder is the limit of the volume of the prism.

The lateral area of the cylinder is the limit of the lateral area of the prism.

The perimeter of a right section of the cylinder is the limit of the perimeter of a right section of the prism.



Let a prism whose base is a regular polygon be inscribed in a given circular cylinder, and a prism whose base is a regular polygon be circumscribed about the cylinder; and let the number of sides of the base of the prism be indefinitely increased.

To prove that the volume of the cylinder is the limit of the volume of the prism, that the lateral area of the cylinder is the limit of the lateral area of the prism, and that the perimeter of a right section of the cylinder is the limit of the perimeter of a right section of the prism.

Proof. If the bases of the prism and cylinder could be made to coincide exactly, the prism and cylinder would coincide exactly; and their volumes would be equal, their lateral areas would be equal, and the perimeters of their right sections would be equal.

We cannot, however, make the bases of the prism and cylinder coincide exactly, and we cannot, therefore, make the prism and cylinder coincide exactly; but by increasing the number of sides of the base of the prism, we can make the base of the prism come as near coinciding with the base of the cylinder as we choose (§ 454), and consequently make the prism come as near coinciding with the cylinder as we choose.

Therefore, the difference between the volumes of the prism and the cylinder can be made less than any assigned value, however small, but cannot be made zero.

The difference between the lateral areas of the prism and cylinder can be made less than any assigned value, however small, but cannot be made zero.

The difference between the perimeters of the right sections of the prism and cylinder can be made less than any assigned value, however small, but cannot be made zero.

Therefore, the volume of the cylinder is the limit of the volume of the prism. § 275

The lateral area of the cylinder is the limit of the lateral area of the prism. § 275

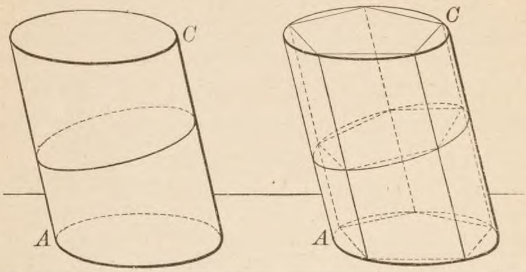
The perimeter of a right section of the cylinder is the limit of the perimeter of a right section of the prism. § 275

Q. E. D.

Note. This theorem can be proved true, when the base of the prism is not a regular polygon and the cylinder is not circular; but it is not the province of Elementary Geometry to treat of cylinders whose bases are not circles.

Proposition XXXIII. Theorem.

696. *The lateral area of a circular cylinder is equal to the product of the perimeter of a right section of the cylinder by an element.*



Let S denote the lateral area, P the perimeter of a right section, and E an element of the cylinder AC.

To prove that $S = P \times E$.

Proof. Inscribe in the cylinder a prism with its base a regular polygon, and denote its lateral area by S', and the perimeter of its right section by P'.

Then $S' = P' \times E$. § 607

If the number of lateral faces of the inscribed prism is indefinitely increased,

S' approaches S as a limit, § 695

and P' approaches P as a limit. § 695

$\therefore P' \times E$ approaches $P \times E$ as a limit. § 279

But $S' = P' \times E$, always. § 607

$\therefore S = P \times E$. § 284

Q. E. D.

697. Cor. 1. *The lateral area of a cylinder of revolution is the product of the circumference of its base by its altitude.*

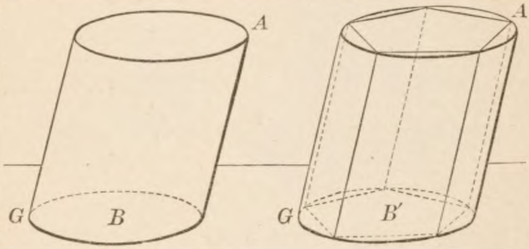
698. Cor. 2. *If* S *denotes the lateral area,* T *the total area,* H *the altitude, and* R *the radius, of a cylinder of revolution,*

$$S = 2\pi R \times H;$$

$$T = 2\pi R \times H + 2\pi R^2 = 2\pi R(H + R).$$

Proposition XXXIV. Theorem.

699. *The volume of a circular cylinder is equal to the product of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude, of the circular cylinder GA.

To prove that $V = B \times H$.

Proof. Inscribe in the cylinder a prism with its base a regular polygon, and denote its volume by V' and its base by B'.

Then $V' = B' \times H$. § 628

If the number of its lateral faces is indefinitely increased,

V' approaches V as a limit, § 695

and B' approaches B as a limit. § 454

$\therefore B' \times H$ approaches $B \times H$ as a limit. § 279

But $V' = B' \times H$, always. § 628

$\therefore V = B \times H$. § 284

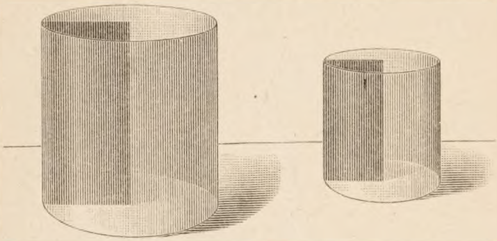
Q. E. D.

700. Cor. *For a cylinder of revolution, with radius* R,

$$V = \pi R^2 \times H.$$

Proposition XXXV. Theorem.

701. *The lateral areas, or the total areas, of similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of their radii; and their volumes are to each other as the cubes of their altitudes, or as the cubes of their radii.*



Let S, S′ denote the lateral areas, T, T′ the total areas, V, V′ the volumes, H, H′ the altitudes, R, R′ the radii, of two similar cylinders of revolution.

To prove that $S : S' = T : T' = H^2 : H'^2 = R^2 : R'^2$,

and $V : V' = H^3 : H'^3 = R^3 : R'^3$.

Proof. Since the generating rectangles are similar, we have by §§ 351, 335,

$$\frac{H}{H'} = \frac{R}{R'} = \frac{H+R}{H'+R'}.$$

Also we have by §§ 698, 700,

$$\frac{S}{S'} = \frac{2\pi RH}{2\pi R'H'} = \frac{R}{R'} \times \frac{H}{H'} = \frac{R^2}{R'^2} = \frac{H^2}{H'^2}.$$

$$\frac{T}{T'} = \frac{2\pi R(H+R)}{2\pi R'(H'+R')} = \frac{R}{R'}\left(\frac{H+R}{H'+R'}\right) = \frac{R^2}{R'^2} = \frac{H^2}{H'^2}.$$

$$\frac{V}{V'} = \frac{\pi R^2 H}{\pi R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{R^3}{R'^3} = \frac{H^3}{H'^3}.$$

Q. E. D.

PROBLEMS OF COMPUTATION.

Ex. 678. The diameter of a well is 6 feet, and the water is 7 feet deep. How many gallons of water are there in the well, reckoning $7\frac{1}{2}$ gallons to the cubic foot?

Ex. 679. When a body is placed under water in a right circular cylinder 60 centimeters in diameter, the level of the water rises 40 centimeters. Find the volume of the body.

Ex. 680. How many cubic yards of earth must be removed in constructing a tunnel 100 yards long, whose section is a semicircle with a radius of 18 feet?

Ex. 681. How many square feet of sheet iron are required to make a funnel 18 inches in diameter and 40 feet long?

Ex. 682. Find the radius of a cylindrical pail 14 inches high that will hold exactly 2 cubic feet.

Ex. 683. The height of a cylindrical vessel that will hold 20 liters is equal to the diameter. Find the altitude and the radius.

Ex. 684. If the total surface of a right circular cylinder is T, and the radius of the base is R, find the height of the cylinder.

Ex. 685. If the lateral surface of a right circular cylinder is S, and the volume is V, find the radius of the base and the height.

Ex. 686. If the circumference of the base of a right circular cylinder is C, and the height H, find the volume V.

Ex. 687. Having given the total surface T of a right circular cylinder, in which the height is equal to the diameter of the base, find the volume V. $V = \frac{T}{3} \cdot \sqrt{\frac{T}{6\pi}}$

Ex. 688. If the circumference of the base of a right circular cylinder is C, and the total surface is T, find the volume V. $V = \frac{2\pi T - C}{4\pi}$

Ex. 689. If the volume of a right circular cylinder is V, and the altitude is H, find the total surface T. $T = 2\pi H \sqrt{\frac{V}{\pi H}} + \frac{2V}{H}$

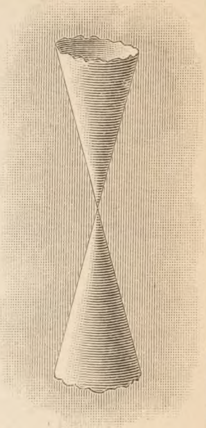
Ex. 690. If V is the volume of a right circular cylinder in which the altitude equals the diameter, find the altitude H, and the total surface T. $H = 2\sqrt[3]{\frac{V}{2\pi}}$

$T = 2\pi \left(\sqrt[3]{\frac{V}{2\pi}}\right)^2 [1+2]$

Ex. 691. If T is the total surface, and H the altitude of a right circular cylinder, find the radius R, and the volume V.

CONES.

702. Def. A **conical surface** is the surface generated by a moving straight line which constantly touches a fixed curve and passes through a fixed point not in the plane of the curve.



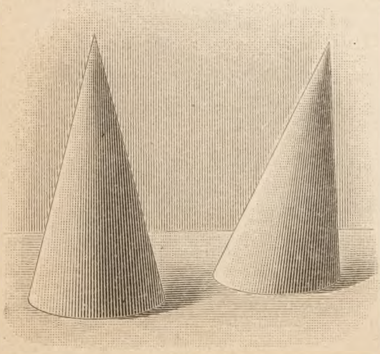
Conical Surface.

The moving straight line which generates the conical surface is called the **generatrix**, the fixed curve the **directrix**, and the fixed point the **vertex**.

703. Def. The generatrix in any position is called an **element of the conical surface.**

If the generatrix is of indefinite length, the surface consists of two portions, one above and the other below the vertex, which are called the **upper and lower nappes**, respectively.

704. Def. If the directrix is a closed curve, the solid bounded by the conical surface and a plane cutting all its elements is called a **cone.**



Cones.

The conical surface is called the **lateral surface of the cone**, and the plane surface is called the **base of the cone.**

The vertex of the conical surface is called the **vertex of the cone**, and the elements of the conical surface are called the **elements of the cone.**

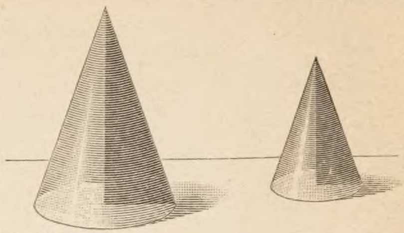
The perpendicular distance from the vertex to the plane of the base is called the **altitude of the cone.**

705. Def. A **circular cone** is a cone whose base is a circle.

The straight line joining the vertex and the centre of the base is called the **axis of the cone.**

If the axis is perpendicular to the base, the cone is called a **right cone.**

If the axis is oblique to the base, the cone is called an **oblique cone.**



Cones of Revolution.

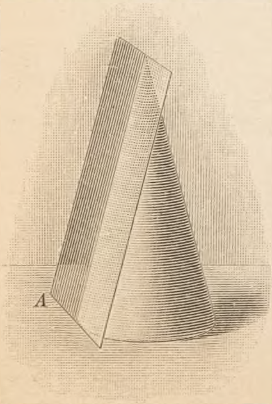
706. Def. A **right circular cone** is a cone whose base is a circle and whose axis is perpendicular to its base.

A right circular cone is called a **cone of revolution,** because it may be generated by the revolution of a right triangle about one of its legs as an axis.

The hypotenuse of the revolving triangle in any position is an element of the surface of the cone, and is called the **slant height** of the cone.

The elements of a cone of revolution are all equal.

707. Def. **Similar cones of revolution** are cones generated by the revolution of similar right triangles about homologous legs.



Tangent Plane.

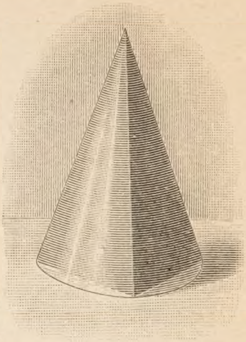
708. Def. A **tangent line to a cone** is a straight line, not an element, which touches the lateral surface of the cone but does not cut it.

709. Def. A **tangent plane to a cone** is a plane which contains an element of the cone but does not cut the surface.

The element contained by the plane is called the **element of contact.**

710. Def. A **pyramid is inscribed in a cone** when its lateral edges are elements of the cone and its base is inscribed in the base of the cone.

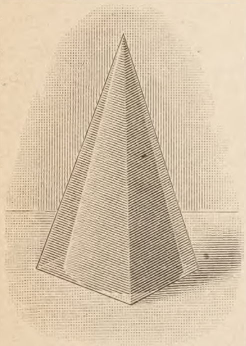
711. Def. A **pyramid is circumscribed about a cone** when its base is circumscribed about the base of the cone and its vertex coincides with the vertex of the cone.



Inscribed Pyramid.

712. Def. A **truncated cone** is the portion of a cone included between the base and a plane cutting all the elements.

A **frustum of a cone** is the portion of a cone included between the base and a plane parallel to the base.



Circumscribed Pyramid.

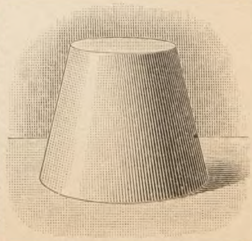
713. Def. The base of the cone is called the **lower base of the frustum**, and the parallel section is called the **upper base of the frustum.**

714. Def. The **altitude of a frustum of a cone** is the perpendicular distance between the planes of its bases.

715. Def. The **lateral surface of a frustum of a cone** is the portion of the lateral surface of the cone included between the bases of the frustum.

716. Def. The elements of a cone between the bases of a frustum of a cone of revolution are equal, and any one is called the **slant height of the frustum.**

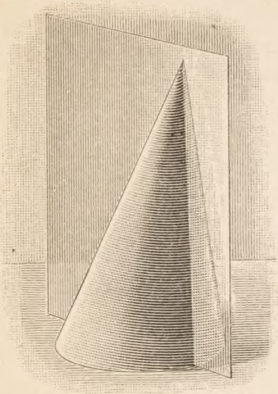
A plane which cuts from the cone a frustum cuts from the inscribed or circumscribed pyramid a frustum.

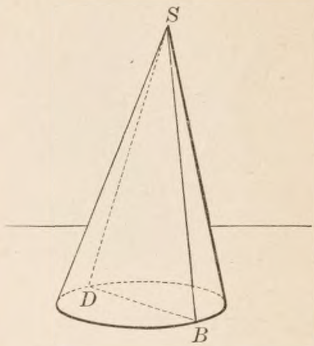


Frustum of a Cone.

Proposition XXXVI. Theorem.

717. *Every section of a cone made by a plane passing through its vertex is a triangle.*





Let SBD be a section of the cone SBD made by a plane passing through the vertex S.

To prove that SBD *is a triangle.*

Proof. BD is a straight line. § 506

Draw the straight lines SB and SD.

These lines are elements of the surface of the cone. § 702

These lines lie in the cutting plane, since their extremities are in the plane. § 492

Hence, SB and SD are the intersections of the conical surface with the cutting plane.

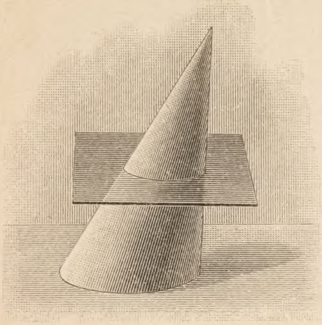
Therefore, the intersections of the conical surface and the plane are straight lines.

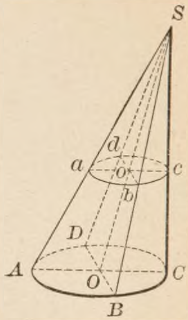
Therefore, the section SBD is a triangle. § 117

Q.E.D.

PROPOSITION XXXVII. THEOREM.

718. *Every section of a circular cone made by a plane parallel to the base is a circle.*





Let the section abcd of the circular cone S-ABCD be parallel to the base.

To prove that abcd is a circle.

Proof. Let O be the centre of the base, and let o be the point in which the axis SO pierces the plane of the parallel section.

Through SO and any elements, SA, SB, pass planes cutting the base in the radii OA, OB, etc., and the section $abcd$ in the straight lines oa, ob, etc.

Then oa and ob are $\parallel$, respectively, to OA and OB. § 528

Therefore, the $\triangle$ Soa and Sob are similar, respectively, to the $\triangle$ SOA and SOB. § 354

$$\therefore \frac{oa}{OA} = \left(\frac{So}{SO}\right) = \frac{ob}{OB}. \quad \text{§ 351}$$

But $OA = OB$. § 217

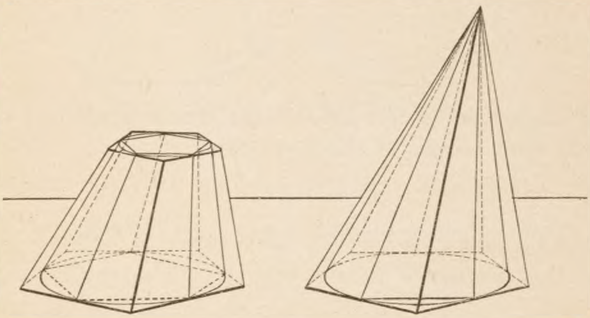
$\therefore oa = ob$, and $abcd$ is a circle. § 216

Q. E. D.

719. COR. *The axis of a circular cone passes through the centre of every section which is parallel to the base.*

Proposition XXXVIII. Theorem.

720. *If a pyramid whose base is a regular polygon is inscribed in or circumscribed about a circular cone, and if the number of sides of the base of the pyramid is indefinitely increased, the volume of the cone is the limit of the volume of the pyramid, and the lateral area of the cone is the limit of the lateral area of the pyramid.*



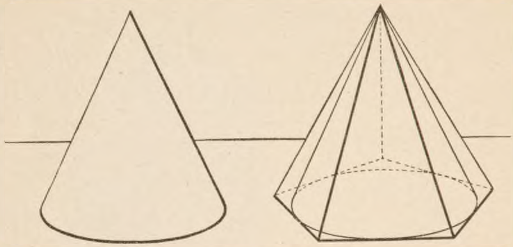
Let a pyramid whose base is a regular polygon be inscribed in a given circular cone, and a pyramid whose base is a regular polygon be circumscribed about the cone, and let the number of sides of the base of the pyramid be indefinitely increased.

The proof is exactly the same as that of Prop. XXXII, if we substitute cone for cylinder and pyramids for prisms.

721. Cor. *The volume of a frustum of a cone is the limit of the volumes of the frustums of the inscribed and circumscribed pyramids, if the number of lateral faces is indefinitely increased, and the lateral area of the frustum of a cone is the limit of the lateral areas of the frustums of the inscribed and circumscribed pyramids.*

PROPOSITION XXXIX. THEOREM.

722. *The lateral area of a cone of revolution is equal to half the product of the slant height by the circumference of the base.*



Let S denote the lateral area, C the circumference of the base, and L the slant height, of the given cone.

To prove that $S = \frac{1}{2} C \times L$.

Proof. Circumscribe about the cone a regular pyramid. Denote the perimeter of its base by P, and its lateral area by S'.

Then $S' = \frac{1}{2} P \times L$. § 643

If the number of the lateral faces of the circumscribed pyramid is indefinitely increased,

S' approaches S as a limit, § 720

and P approaches C as a limit. § 454

$\therefore \frac{1}{2} P \times L$ approaches $\frac{1}{2} C \times L$ as a limit. § 279

But $S' = \frac{1}{2} P \times L$, always. § 643

$\therefore S = \frac{1}{2} C \times L$. § 284

Q.E.D.

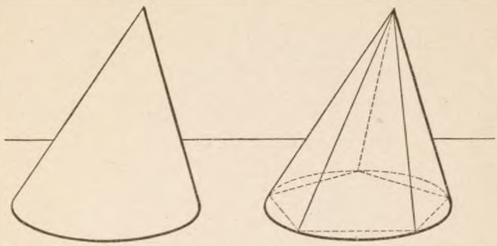
723. COR. *If* S *denotes the lateral area,* T *the total area,* H *the altitude,* R *the radius of the base, of a cone of revolution,*

$$S = \frac{1}{2}(2\pi R \times L) = \pi RL;$$

$$T = \pi RL + \pi R^2 = \pi R(L + R).$$

Proposition XL. Theorem.

724. *The volume of a circular cone is equal to one third the product of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude of the given cone.

To prove that $V = \frac{1}{3} B \times H.$

Proof. Inscribe in the cone a pyramid with a regular polygon for its base.

Denote its volume by V', and its base by B'.

Then $V' = \frac{1}{3} B' \times H.$ § 652

If the number of the lateral faces of the inscribed pyramid is indefinitely increased,

V' approaches V as a limit, § 720

and B' approaches B as a limit. § 454

$\therefore \frac{1}{3} B' \times H$ approaches $\frac{1}{3} B \times H$ as a limit. § 279

But $V' = \frac{1}{3} B' \times H$, always. § 652

$\therefore V = \frac{1}{3} B \times H.$ § 284

Q. E. D.

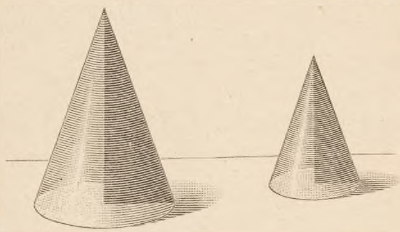
725. Cor. *If* R *is the radius of the base*,

$B = \pi R^2.$ § 463

$\therefore V = \frac{1}{3} \pi R^2 \times H.$

Proposition XLI. Theorem.

726. *The lateral areas, or the total areas, of two similar cones of revolution are to each other as the squares of their altitudes, as the squares of their radii, or as the squares of their slant heights; and their volumes are to each other as the cubes of their altitudes, as the cubes of their radii, or as the cubes of their slant heights.*



Let S and S′ denote the lateral areas, T and T′ the total areas, V and V′ the volumes, H and H′ the altitudes, R and R′ the radii, L and L′ the slant heights, of two similar cones of revolution.

To prove that $S : S' = T : T' = H^2 : H'^2 = R^2 : R'^2 = L^2 : L'^2$,

and $V : V' = H^3 : H'^3 = R^3 : R'^3 = L^3 : L'^3$.

Proof. $$\frac{H}{H'} = \frac{R}{R'} = \frac{L}{L'} = \frac{L+R}{L'+R'}.$$ §§ 351, 335

$$\frac{S}{S'} = \frac{\pi RL}{\pi R'L'} = \frac{R}{R'} \times \frac{L}{L'} = \frac{R^2}{R'^2} = \frac{L^2}{L'^2} = \frac{H^2}{H'^2},$$ § 723

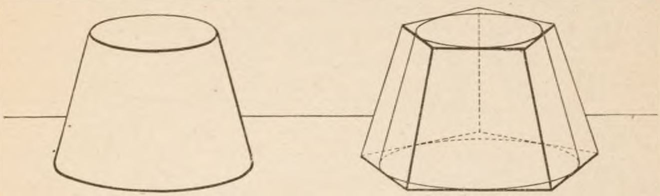
$$\frac{T}{T'} = \frac{\pi R(L+R)}{\pi R'(L'+R')} = \frac{R}{R'} \times \frac{L+R}{L'+R'} = \frac{R^2}{R'^2} = \frac{L^2}{L'^2} = \frac{H^2}{H'^2}.$$

$$\frac{V}{V'} = \frac{\frac{1}{3}\pi R^2 H}{\frac{1}{3}\pi R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{R^3}{R'^3} = \frac{H^3}{H'^3} = \frac{L^3}{L'^3}.$$ § 725

Q. E. D.

PROPOSITION XLII. THEOREM.

727. *The lateral area of a frustum of a cone of revolution is equal to half the sum of the circumferences of its bases multiplied by the slant height.*



Let S denote the lateral area, C and c the circumferences of its bases, R and r their radii, and L the slant height.

To prove that $S = \frac{1}{2}(C + c) \times L$.

Proof. Circumscribe about the frustum of the cone a frustum of a regular pyramid. Denote the lateral area of this frustum by S', the perimeters of its lower and upper bases by P and p, respectively, and its slant height by L.

Then $S' = \frac{1}{2}(P + p) \times L$. § 644

If the number of lateral faces is indefinitely increased,

S' approaches S as a limit, § 721

and $P + p$ approaches $C + c$ as a limit. §§ 454, 278

$\therefore \frac{1}{2}(P + p)L$ approaches $\frac{1}{2}(C + c)L$ as a limit. § 279

But $S' = \frac{1}{2}(P + p) \times L$, always. § 644

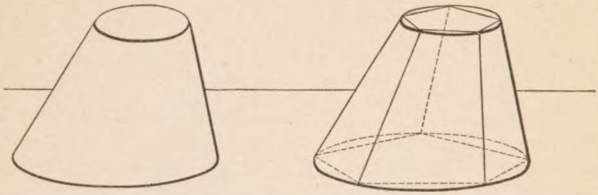
$\therefore S = \frac{1}{2}(C + c) \times L$. § 284

Q.E.D.

728. COR. *The lateral area of a frustum of a cone of revolution is equal to the circumference of a section equidistant from its bases multiplied by its slant height.*

Proposition XLIII. Theorem.

729. *The volume of a frustum of a circular cone is equivalent to the sum of the volumes of three cones whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and the mean proportional between the bases of the frustum.*



Let V denote the volume, B the lower base, b the upper base, H the altitude of a frustum of a circular cone.

To prove that $V = \frac{1}{3} H(B + b + \sqrt{B \times b})$.

Proof. Let V' denote the volume, B' and b' the lower and upper bases, and H the altitude, of an inscribed frustum of a pyramid with regular polygons for its bases.

Then $V' = \frac{1}{3} H(B' + b' + \sqrt{B' \times b'})$. § 657

If the number of the lateral faces of the inscribed frustum is indefinitely increased,

V' approaches V as a limit, § 721

B' approaches B as a limit, § 454

and b' approaches b as a limit. § 454

$\therefore B' \times b'$ approaches $B \times b$ as a limit. § 281

$\therefore \sqrt{B' \times b'}$ approaches $\sqrt{B \times b}$ as a limit. § 283

$\therefore B' + b' + \sqrt{B' \times b'}$ approaches

$B + b + \sqrt{B \times b}$ as a limit. § 278

But $\quad V' = \frac{1}{3} H(B' + b' + \sqrt{B' \times b'})$, always. § 657

$\therefore V = \frac{1}{3} H(B + b + \sqrt{B \times b})$. § 284

Q.E.D.

730. Cor. *If the frustum is that of a cone of revolution, and* R *and* r *are the radii of its bases,*

$$B = \pi R^2,\ b = \pi r^2. \qquad \text{§ 463}$$

$$\therefore \sqrt{B \times b} = \sqrt{\pi R^2 \times \pi r^2} = \pi R r.$$

$$\therefore V = \tfrac{1}{3} \pi H(R^2 + r^2 + Rr).$$

Ex. 692. The radii of the bases of the frustum of a right circular cone are 20 inches and 13 inches, respectively. If the altitude of the frustum is 15 inches and is bisected by a plane parallel to the bases, what is the lateral area of each frustum made by the plane?

Ex. 693. The radius of the base of a right circular cone is 8 feet, and the altitude is 10 feet. Find the area of its lateral surface, the area of its total surface, and the volume. $L = 327,816$ $T = 438,776$ $V = 337,3068$

Ex. 694. The height of a right circular cone is equal to the diameter of its base. Find the ratio of the area of the base to the area of the lateral surface. $= R\frac{\sqrt{5}}{2}$

Ex. 695. The slant height of a right circular cone is 2 feet. At what distance from the vertex must the slant height be cut by a plane parallel to the base, in order that the lateral surface may be divided into two equivalent parts? At the distance $\sqrt{2}$ feet

Ex. 696. What does the volume V of a circular cone become, if the altitude is doubled? If the radius of the base is doubled? If both the altitude and the radius of the base are doubled? (I) 8 (II) 8 (III) 64

Ex. 697. The slant height L of a right circular cone is equal to the diameter of the base. Find the total surface T. $T = \frac{3}{2}\pi L^2$

Ex. 698. If T is the total surface of a right circular cone whose slant height equals the diameter of the base, find the volume V. $V = \frac{T}{18}\sqrt{\frac{3T}{6\pi}}$

Ex. 699. If T is the total surface of a right circular cone, and R is the radius of the base, find the volume V. $V = \frac{\pi R^2}{3}\sqrt{\frac{T^2 - 2T}{\pi}}$

Ex. 700. If T is the total surface of a right circular cone, and S is the lateral surface, find the volume V. $V = \frac{1}{3} H \sqrt{(T-S)\pi}$

THE PRISMATOID FORMULA.

731. Def. A polyhedron is called a **prismatoid** if it has for bases two polygons in parallel planes, and for lateral faces triangles or trapezoids with one side common with one base and the opposite vertex or side common with the other base.

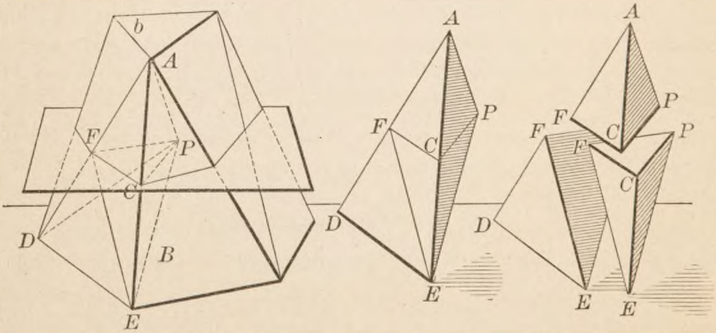
732. Def. The **altitude** of a prismatoid is the perpendicular distance between the planes of its bases.

The **mid-section** of a prismatoid is the section made by a plane parallel to its bases and midway between them.

The mid-section bisects the altitude and all the lateral edges.

Proposition XLIV. Theorem.

733. *The volume of a prismatoid is equal to the product of one sixth of its altitude into the sum of its bases and four times its mid-section.*



Let V denote the volume, B and b the bases, M the mid-section, and H the altitude, of a given prismatoid.

To prove that $V = \frac{1}{6} H(B + b + 4M)$.

Proof. If any lateral face is a trapezoid, divide it into two triangles by a diagonal.

Take any point P in the mid-section and join P to the vertices of the polyhedron and of the mid-section.

Divide the prismatoid into pyramids which have their vertices at P, and for their respective bases the lower base B, the upper base b, and the lateral faces of the prismatoid.

The *lateral* pyramid $P\text{-}ADE$ is composed of three pyramids $P\text{-}AFC$, $P\text{-}FCE$, and $P\text{-}FDE$.

Now $P\text{-}AFC$ may be regarded as having vertex A and base PFC, and $P\text{-}FCE$, as having vertex E and base PFC.

Hence, the volume of $P\text{-}AFC$ is equal to $\frac{1}{6}H \times PFC$,

and the volume of $P\text{-}FCE$ is equal to $\frac{1}{6}H \times PFC$. § 651

The pyramid $P\text{-}FDE$ is equivalent to twice $P\text{-}FCE$.

For they have the same vertex P, and the base FDE is twice the base FCE, since the $\triangle FDE$ has its base DE twice the base FC of the $\triangle FCE$ (§ 405), and these triangles have the same altitude.

Hence, the volume of $P\text{-}FDE$ is equal to $\frac{2}{6}H \times PFC$.

Therefore, the volume of $P\text{-}ADE$, which is composed of $P\text{-}AFC$, $P\text{-}FCE$, and $P\text{-}FDE$, is equal to $\frac{4}{6}H \times PFC$.

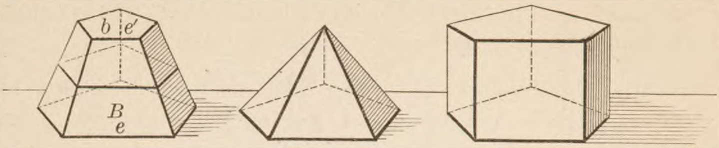
In like manner, the volume of each lateral pyramid is equal to $\frac{4}{6}H \times$ the area of that part of the mid-section which is included within it; and, therefore, the total volume of all these lateral pyramids is equal to $\frac{4}{6}H \times M$.

The volume of the pyramid with base B is $\frac{1}{6}H \times B$,

and the volume of the pyramid with base b is $\frac{1}{6}H \times b$. § 651

Therefore, $V = \frac{1}{6}H(B + b + 4M)$. Q.E.D.

734. *The prismatoid formula may be used for finding the volumes of all the solids of Elementary Geometry:*



The formula for the volume of the frustum of a pyramid is

$$V = \tfrac{1}{3} H(B + b + \sqrt{B \times b}). \qquad \text{§ 657}$$

But $\tfrac{1}{3} H(B + b + \sqrt{B \times b}) = \tfrac{1}{6} H(B + b + 4M)$. (1)

For if e and e' are corresponding sides of B and b, then $\tfrac{1}{2}(e + e')$ is the corresponding side of M. § 190

Therefore, $\dfrac{e}{\tfrac{1}{2}(e+e')} = \dfrac{\sqrt{B}}{\sqrt{M}}$, and $\dfrac{e'}{\tfrac{1}{2}(e+e')} = \dfrac{\sqrt{b}}{\sqrt{M}}$. § 414

Adding and reducing, $\dfrac{2}{1} = \dfrac{\sqrt{B} + \sqrt{b}}{\sqrt{M}}$.

Therefore, $2\sqrt{M} = \sqrt{B} + \sqrt{b}$.

Squaring, $4M = B + b + 2\sqrt{B \times b}$.

If we put this value of $4M$ in (1), the two members become identically equal.

735. If the base b becomes zero, we have a pyramid, and the prismatoid formula becomes $V = \tfrac{1}{3} H \times B$. § 652

736. If the bases B and b are equal, we have a prism, and the prismatoid formula becomes $V = H \times B$. § 628

NOTE. The Prismatoid Formula is taken by permission from the work on Mensuration by Dr. George Bruce Halsted, Professor of Mathematics in the University of Texas.

Ex. 701. Show that the prismatoid formula can be used for finding the volume of the frustum of a cone, for finding the volume of a cone, for finding the volume of a cylinder.

FRUSTUMS OF PYRAMIDS AND OF CONES.

Ex. 702. How many square feet of tin are required to make a funnel, if the diameters of the top and bottom are 28 inches and 14 inches, respectively, and the height is 24 inches? π. 14. 75

Ex. 703. Find the expense, at 60 cents a square foot, of polishing the curved surface of a marble column in the shape of the frustum of a right circular cone whose slant height is 12 feet, and the radii of whose bases are 3 feet 6 inches and 2 feet 4 inches, respectively. the expense = 13188 cents

Ex. 704. The slant height of the frustum of a regular pyramid is 20 feet; the sides of its square bases 40 feet and 16 feet. Find the volume. 13312 c. f

Ex. 705. If the bases of the frustum of a pyramid are regular hexagons whose sides are 1 foot and 2 feet, respectively, and the volume of the frustum is 12 cubic feet, find the altitude.

Ex. 706. The frustum of a right circular cone 14 feet high has a volume of 924 cubic feet. Find the radii of its bases if their sum is 9 feet.

Ex. 707. From a right circular cone whose slant height is 30 feet, and the circumference of whose base is 10 feet, there is cut off by a plane parallel to the base a cone whose slant height is 6 feet. Find the lateral area and the volume of the frustum. $L = 144$ square feet

Ex. 708. Find the difference between the volume of the frustum of a pyramid whose bases are squares, 8 feet and 6 feet, respectively, on a side and the volume of a prism of the same altitude whose base is a section of the frustum parallel to its bases and equidistant from them. $\frac{1}{3}H$

Ex. 709. A Dutch windmill in the shape of the frustum of a right cone is 12 meters high. The outer diameters at the bottom and the top are 16 meters and 12 meters, the inner diameters 12 meters and 10 meters. How many cubic meters of stone were required to build it? 734,76 cubic meters

Ex. 710. The chimney of a factory has the shape of a frustum of a regular pyramid. Its height is 180 feet, and its upper and lower bases are squares whose sides are 10 feet and 16 feet, respectively. The flue is throughout a square whose side is 7 feet. How many cubic feet of material does the chimney contain? = 27720

Ex. 711. Find the volume V of the frustum of a cone of revolution, having given the slant height L, the height H, and the lateral area S.

$V = \frac{HS}{L}$

EQUIVALENT SOLIDS.

Ex. 712. A cube each edge of which is 12 inches is transformed into a right prism whose base is a rectangle 16 inches long and 12 inches wide. Find the height of the prism, and the difference between its total area and the total area of the cube. $H = 9$ the difference = 24 square feet

Ex. 713. The dimensions of a rectangular parallelopiped are a, b, c. Find (i) the height of an equivalent right circular cylinder, having a for the radius of its base; (ii) the height of an equivalent right circular cone having a for the radius of its base. (i) $\frac{bc}{\pi a}$ (ii) $\frac{3bc}{\pi a}$

Ex. 714. A regular pyramid 12 feet high is transformed into a regular prism with an equivalent base. Find the height of the prism. 4 feet

Ex. 715. The diameter of a cylinder is 14 feet, and its height is 8 feet. Find the height of an equivalent right prism, the base of which is a square with a side 4 feet long. $H = \pi \cdot 24{,}5$

Ex. 716. If one edge of a cube is a, what is the height H of an equivalent right circular cylinder whose radius is R? $H = \frac{a^3}{\pi R^2}$

Ex. 717. The heights of two equivalent right circular cylinders are in the ratio 4 : 9. If the diameter of the first is 6 feet, what is the diameter of the other?

Ex. 718. A right circular cylinder 6 feet in diameter is equivalent to a right circular cone 7 feet in diameter. If the height of the cone is 8 feet, what is the height of the cylinder? = 3,66

Ex. 719. The frustum of a regular pyramid 6 feet high has for bases squares 5 feet and 8 feet on a side. Find the height of an equivalent regular pyramid whose base is a square 12 feet on a side. $H = 5{,}38$

Ex. 720. The frustum of a cone of revolution is 5 feet high, and the diameters of its bases are 2 feet and 3 feet, respectively. Find the height of an equivalent right circular cylinder whose base is equal in area to the section of the frustum made by a plane parallel to its bases and equidistant from the bases. $H = 6{,}3$

Ex. 721. Find the edge of a cube equivalent to a regular tetrahedron whose edge measures 3 inches. $\sqrt[3]{4{,}4252}$

Ex. 722. Find the edge of a cube equivalent to a regular octahedron whose edge measures 3 inches. $\sqrt[3]{12{,}72}$

SIMILAR SOLIDS.

they are 6,36 | 4,7 | 3,1

Ex. 723. The dimensions of a trunk are 4 feet, 3 feet, 2 feet. Find the dimensions of a trunk similar in shape that will hold four times as much.

Ex. 724. By what number must the dimensions of a cylinder be multiplied to obtain a similar cylinder (i) whose surface shall be n times that of the first; (ii) whose volume shall be n times that of the first? by n

Ex. 725. A pyramid is cut by a plane parallel to the base which passes midway between the vertex and the plane of the base. Compare the volumes of the entire pyramid and the pyramid cut off. the first 8 times the second

Ex. 726. The height of a regular hexagonal pyramid is 36 feet, and one side of the base is 6 feet. What are the dimensions of a similar pyramid whose volume is $\frac{1}{20}$ that of the first? the side of the base $\sqrt[3]{10,8}$ and $H = \sqrt[3]{64,8}$

Ex. 727. The length of one of the lateral edges of a pyramid is 4 meters. How far from the vertex will this edge be cut by a plane parallel to the base, which divides the pyramid into two equivalent parts? $= \sqrt[3]{8}$

Ex. 728. A lateral edge of a pyramid is a. At what distances from the vertex will this edge be cut by two planes parallel to the base that divide the pyramid into three equivalent parts? At $\frac{a}{\sqrt[3]{3}}$ and at $a\sqrt[3]{\frac{2}{3}}$

Ex. 729. A lateral edge of a pyramid is a. At what distance from the vertex will this edge be cut by a plane parallel to the base that divides the pyramid into two parts which are to each other as 3 : 4? At $a\sqrt[3]{\frac{3}{7}}$

Ex. 730. The volumes of two similar cones are 54 cubic feet and 432 cubic feet. The height of the first is 6 feet; what is the height of the other?

Ex. 731. Two right circular cylinders have their diameters equal to their heights. Their volumes are as 3 : 4. Find the ratio of their heights. $= \sqrt[3]{\frac{3}{4}}$

Ex. 732. Find the dimensions of a right circular cylinder $\frac{15}{16}$ as large as a similar cylinder whose height is 20 feet, and diameter 10 feet. $H = \sqrt[3]{7500}$ $R = \sqrt[3]{117,18}$

Ex. 733. The height of a cone of revolution is H, and the radius of its base is R. Find the dimensions of a similar cone three times as large. $h = H\sqrt[3]{3}$ $r = R\sqrt[3]{3}$

Ex. 734. The height of the frustum of a right cone is $\frac{2}{5}$ the height of the entire cone. Compare the volumes of the frustum and the cone. the cone is 1,27 the frustum

Ex. 735. The frustum of a pyramid is 8 feet high, and two homologous edges of its bases are 4 feet and 3 feet, respectively. Compare the volume of the frustum and that of the entire pyramid. the first is $\frac{37}{64}$ times the second

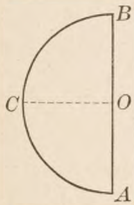
BOOK VIII.

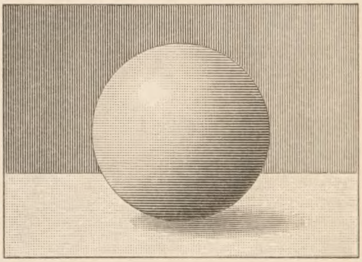
THE SPHERE.

PLANE SECTIONS AND TANGENT PLANES.

737. DEF. A **sphere** is a solid bounded by a surface all points of which are equally distant from a point within called the **centre**.

738. A sphere may be generated by the revolution of a semicircle ACB about its diameter AB as an axis.





739. DEF. A **radius of a sphere** is a straight line drawn from the centre to the surface.

A **diameter of a sphere** is a straight line passing through the centre and limited by the surface.

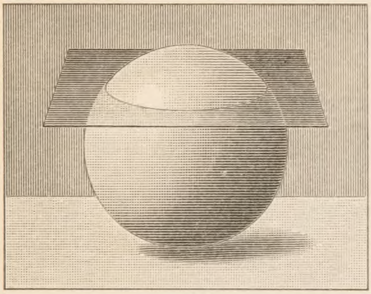
740. *All the radii of a sphere are equal, and all the diameters of a sphere are equal.*

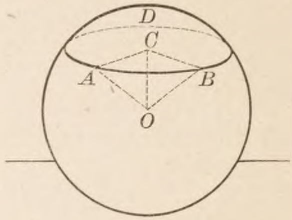
741. DEF. A **line or plane is tangent to a sphere** when it has one, and only one, point in common with the surface of the sphere.

742. DEF. **Two spheres are tangent to each other** when their surfaces have one, and only one, point in common.

PROPOSITION I. THEOREM.

743. *Every section of a sphere made by a plane is a circle.*





Let O be the centre of the sphere, and ABD any section made by a plane.

To prove that the section ABD is a circle.

Proof. Draw the radii OA, OB, to any two points A, B, in the boundary of the section, and draw $OC \perp$ to the section.

The rt. $\triangle$ OAC and OBC are equal. § 151

For OC is common, and $OA = OB$. § 740

$\therefore CA = CB$. § 128

But A and B are any two points in the boundary of the section; hence, all points in the boundary are equally distant from C, and the section ABD is a circle. § 216

Q. E. D.

744. COR. 1. *The line joining the centre of a sphere to the centre of a circle of the sphere is perpendicular to the plane of the circle.*

745. COR. 2. *Circles of a sphere made by planes equally distant from the centre are equal.*

For $\overline{AC}^2 = \overline{AO}^2 - \overline{OC}^2$; and AO and OC are the same for all equally distant circles; therefore, AC is the same.

746. Cor. 3. *Of two circles made by planes unequally distant from the centre, the nearer is the greater.*

747. Def. A **great circle of a sphere** is a section made by a plane which passes through the centre of the sphere.

748. Def. A **small circle of a sphere** is a section made by a plane which does not pass through the centre of the sphere.

749. Def. The **axis of a circle of a sphere** is the diameter of the sphere which is perpendicular to the plane of the circle.

The ends of the axis are called the **poles of the circle.**

750. Cor. 1. *Parallel circles have the same axis and the same poles.*

751. Cor. 2. *All great circles of a sphere are equal.*

752. Cor. 3. *Every great circle bisects the sphere.*

For the two parts into which the sphere is divided can be so placed that they will coincide; otherwise there would be points on the surface unequally distant from the centre.

753. Cor. 4. *Two great circles bisect each other.*

For the intersection of their planes passes through the centre, and is, therefore, a diameter of each circle.

754. Cor. 5. *If the planes of two great circles are perpendicular, each circle passes through the poles of the other.*

755. Cor. 6. *Through two given points on the surface of a sphere an arc of a great circle may always be drawn.*

For the two given points together with the centre of the sphere determine the plane of a great circle which passes through the two given points. § 496

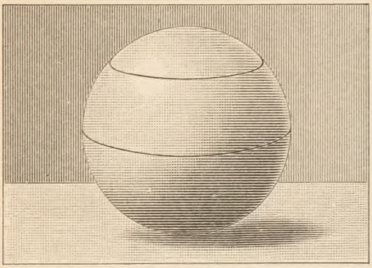
If, however, the two given points are the ends of a diameter, the position of the circle is not determined. § 494

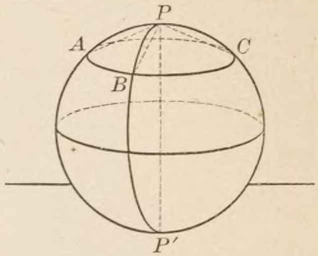
756. Cor. 7. *Through three given points on the surface of a sphere one circle may be drawn, and only one.* § 496

757. DEF. The **distance between two points on the surface of a sphere** is the arc of the great circle that joins them.

PROPOSITION II. THEOREM.

758. *The distances of all points in the circumference of a circle of a sphere from its poles are equal.*





Let P, P′ be the poles of the circle ABC, and A, B, C, any points in its circumference.

To prove that the great circle arcs PA, PB, PC are equal.

Proof. The straight lines PA, PB, PC are equal. § 514

Therefore, the arcs PA, PB, PC are equal. § 241

In like manner, the great circle arcs $P'A$, $P'B$, $P'C$ may be proved equal. Q.E.D.

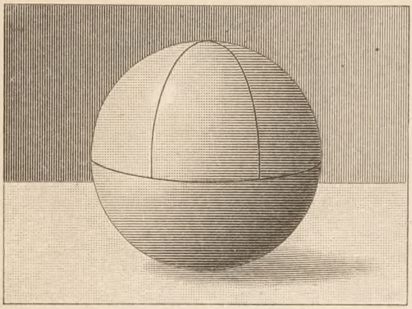
759. DEF. The distance on the surface of the sphere from the nearer pole of a small circle to any point in the circumference of the circle is called the **polar distance of the circle.**

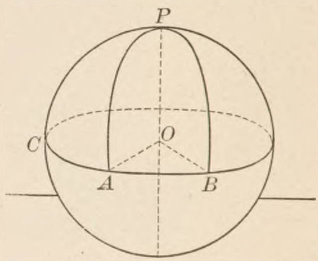
760. DEF. The distance on the surface of the sphere of a *great circle from either of its poles* is called the polar distance of the circle.

761. COR. *The polar distance of a great circle is a quadrant; that is, one fourth the circumference of a great circle.*

PROPOSITION III. THEOREM.

762. *A point on the surface of a sphere, which is at the distance of a quadrant from each of two other points, not the extremities of a diameter, is a pole of the great circle passing through these points.*





Let the distances PA and PB be quadrants, and let ABC be the great circle passing through A and B.

To prove that P is a pole of the great circle ABC.

Proof. The $\angle s$ POA and POB are rt. $\angle s$. § 288

$\therefore PO$ is $\perp$ to the plane of the $\odot$ ABC. § 507

Hence, P is a pole of the $\odot$ ABC. § 749

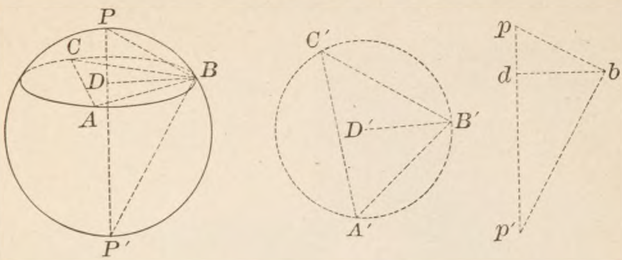
Q.E.D.

763. SCHOLIUM. The above theorem enables us to describe with the compasses an arc of a great circle through two given points of the surface of a sphere. For, if with A and B as centres, and an opening of the compasses equal to the *chord* of a quadrant of a great circle, we describe arcs, these arcs will intersect at a point P. Then, with P as centre, the arc passing through A and B may be described.

In order to make the opening of the compasses equal to the chord of a quadrant of a great circle, the radius of the sphere must be known.

PROPOSITION IV. PROBLEM.

764. *Given a material sphere to find its diameter.*



Let PBP′C represent a material sphere.

It is required to find its diameter.

From any point P of the given surface describe a circumference ABC on the surface.

Then the straight line PB is known.

Take three points A, B, and C in this circumference, and with the compasses measure the chords AB, BC, and CA.

Construct the $\triangle A'B'C'$, with sides equal respectively to AB, BC, and CA, and circumscribe a $\odot$ about the $\triangle$.

The radius $D'B'$ of this $\odot$ is equal to the radius of $\odot ABC$.

Construct the rt. $\triangle bdp$, having the hypotenuse bp equal to BP, and one side bd equal to $B'D'$.

Draw $bp' \perp$ to bp, meeting pd produced in p'.

Then pp' is equal to the diameter of the given sphere.

Proof. Suppose the diameter PP' and the straight line $P'B$ drawn.

The $\triangle$s BDP and bdp are equal. § 151

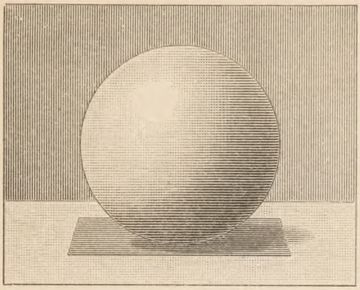
Hence, the $\triangle$s PBP' and pbp' are equal. § 142

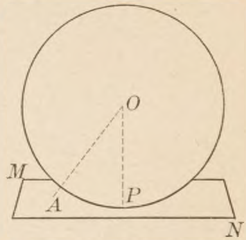
Therefore, $pp' = PP'$. § 128

Q. E. F.

PROPOSITION V. THEOREM.

765. *A plane perpendicular to a radius at its extremity is tangent to the sphere.*





Let O be the centre of a sphere, and MN a plane perpendicular to the radius OP, at its extremity P.

To prove that MN is tangent to the sphere.

Proof. Let A be any point except P in MN. Draw OA.

Then $OP < OA$. § 512

Therefore, the point A is without the sphere. § 737

But A is any point, except P, in the plane MN.

$\therefore$ every point in MN, except P, is without the sphere.

Therefore, MN is tangent to the sphere at P. § 741

Q. E. D.

766. Cor. 1. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

767. Cor. 2. *A line tangent to a circle of a sphere lies in the plane tangent to the sphere at the point of contact.* § 508

768. Cor. 3. *A line in a tangent plane drawn through the point of contact is tangent to the sphere at that point.*

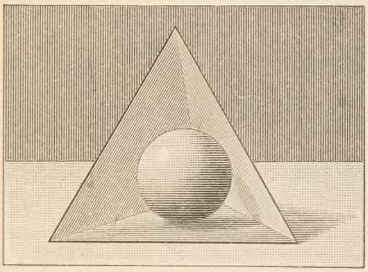
769. Cor. 4. *The plane of two lines tangent to a sphere at the same point is tangent to the sphere at that point.*

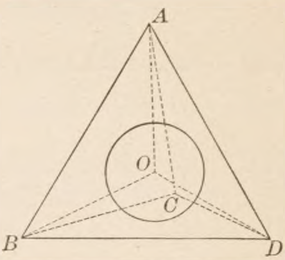
770. Def. **A sphere is inscribed in a polyhedron** when all the faces of the polyhedron are tangent to the sphere.

771. Def. **A sphere is circumscribed about a polyhedron** when all the vertices of the polyhedron lie in the surface of the sphere.

Proposition VI. Theorem.

772. *A sphere may be inscribed in any given tetrahedron.*





Let A-BCD be the given tetrahedron.

To prove that a sphere may be inscribed in A-BCD.

Proof. Bisect the dihedral ∠s at the edges AB, BC, and AC by the planes OAB, OBC, and OAC, respectively.

Every point in the plane OAB is equally distant from the faces ABC and ABD. § 559

For a like reason, every point in the plane OBC is equally distant from the faces ABC and DBC; and every point in the plane OAC is equally distant from the faces ABC and ADC.

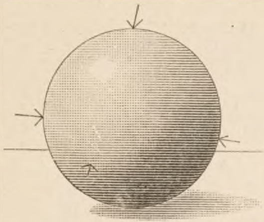
Therefore, O, the common intersection of these three planes, is equally distant from the four faces of the tetrahedron, and is the centre of the sphere inscribed in the tetrahedron. § 770

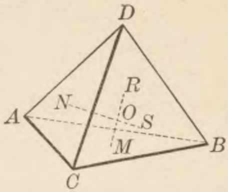
Q.E.D.

773. Cor. *The six planes which bisect the six dihedral angles of a tetrahedron intersect in the same point.*

PROPOSITION VII. THEOREM.

774. *A sphere may be circumscribed about any given tetrahedron.*





Let D-ABC be the given tetrahedron.

To prove that a sphere may be circumscribed about D-ABC.

Proof. Let M, N, respectively, be the centres of the circles circumscribed about the faces ABC, ACD.

Let MR be $\perp$ to the face ABC, NS $\perp$ to the face ACD.

Then MR is the locus of points equidistant from A, B, C, and NS is the locus of points equidistant from A, C, D. § 516

Therefore, MR and NS lie in the same plane, the plane $\perp$ to AC at its middle point. § 517

Also MR and NS, being $\perp$ to planes which are not $\parallel$, cannot be $\parallel$, and must therefore meet at some point O.

$\therefore$ O is equidistant from A, B, C, and D.

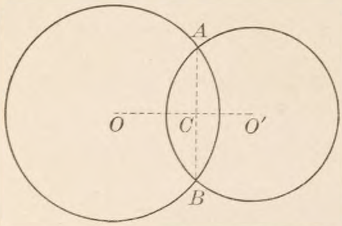
Therefore, a spherical surface whose centre is O, and radius OA, will pass through the points A, B, C, and D. Q.E.D.

775. COR. 1. *The four perpendiculars erected at the centres of the faces of a tetrahedron meet at the same point.*

776. COR. 2. *The six planes perpendicular to the edges of a tetrahedron at their middle points intersect at the same point.*

Proposition VIII. Theorem.

777. *The intersection of two spherical surfaces is the circumference of a circle whose plane is perpendicular to the line joining the centres of the surfaces and whose centre is in that line.*



Let O, O′ be the centres of the spherical surfaces, and let a plane passing through O, O′ cut the spheres in great circles whose circumferences intersect in the points A and B.

To prove that the spherical surfaces intersect in the circumference of a circle whose plane is perpendicular to OO', and whose centre is the point C where AB meets OO'.

Proof. The common chord AB is $\perp$ to OO' and bisected at C. § 264

If the plane of the two great circles is revolved about OO' as an axis, their circumferences will generate the two spherical surfaces, and the point A will describe the line of intersection of the surfaces.

But during the revolution AC will remain constant in length and $\perp$ to OO'.

Therefore, the line of intersection described by the point A will be the circumference of a circle whose centre is C and whose plane is $\perp$ to OO'. § 508

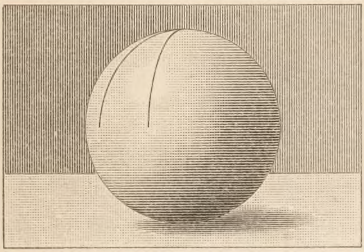
Q. E. D.

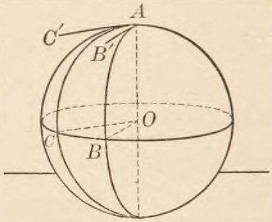
FIGURES ON THE SURFACE OF A SPHERE.

778. Def. The **angle of two curves** passing through the same point is the angle formed by two tangents to the curves at that point. The angle formed by the intersection of two arcs of great circles of a sphere is called a **spherical angle.**

Proposition IX. Theorem.

779. *A spherical angle is measured by the arc of the great circle described from its vertex as a pole and included between its sides (produced if necessary).*





Let AB, AC be arcs of great circles intersecting at A; AB′ and AC′, the tangents to these arcs at A; BC the arc of the great circle described from A as a pole and included between AB and AC.

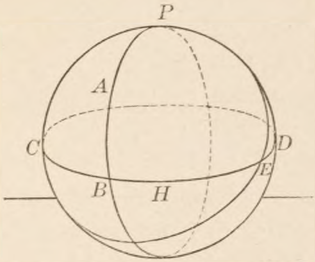
To prove that the spherical $\angle BAC$ *is measured by arc* BC.

Proof.	In the plane AOB, AB' is $\perp$ to AO,	§ 254
	and OB is $\perp$ to AO.	§ 288
	$\therefore AB'$ is $\parallel$ to OB.	§ 104
Similarly,	AC' is $\parallel$ to OC.	
	$\therefore \angle B'AC' = \angle BOC$.	§ 534
But	$\angle BOC$ is measured by arc BC.	§ 288
	$\therefore \angle B'AC'$ is measured by arc BC.	
	$\therefore \angle BAC$ is measured by arc BC.	Q. E. D.

780. Cor. *A spherical angle has the same measure as the dihedral angle formed by the planes of the two circles.*

Proposition X. Problem.

781. *To describe an arc of a great circle through a given point perpendicular to a given arc of a great circle.*



Let A be a point on the surface of a sphere, CHD an arc of a great circle, P its pole.

From A as a pole describe an arc of a great circle cutting CHD at E.

From E as a pole describe the arc AB through A.

Then AB is the arc required.

Proof. The arc AB is the arc of a great circle, and E is its pole by construction. § 762

The point E is at the distance of a quadrant from P. § 761

Therefore, the arc AB produced will pass through P.

Since the spherical $\angle PBE$ is measured by the arc PE of a great circle (§ 779), the $\angle ABE$ is a right angle.

Therefore, the arc AB is $\perp$ to the arc CHD. Q.E.F.

Ex. 736. Every point in a great circle which bisects a given arc of a great circle at right angles is equidistant from the extremities of the given arc.

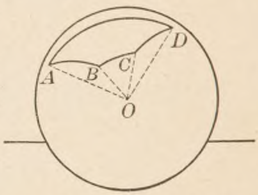
782. Def. A **spherical polygon** is a portion of the surface of a sphere bounded by three or more arcs of great circles.

The bounding arcs are the **sides** of the polygon; the angles between the sides are the **angles** of the polygon; the points of intersection of the sides are the **vertices** of the polygon.

The values of the sides of a spherical polygon are usually expressed in degrees, minutes, and seconds.

783. The planes of the sides of a spherical polygon form a polyhedral angle whose vertex is the centre of the sphere, whose face angles are measured by the sides of the polygon, and whose dihedral angles have the same numerical measure as the angles of the polygon.

Thus, the planes of the sides of the polygon $ABCD$ form the polyhedral angle O-$ABCD$. The face angles AOB, BOC, etc., are measured by the sides AB, BC, etc., of the polygon. The dihedral angle whose edge is OA has the same measure as the spherical angle BAD, etc. Hence,



784. *From any property of polyhedral angles we may infer an analogous property of spherical polygons; and conversely.*

785. Def. A **spherical polygon is convex** if the corresponding polyhedral angle is convex (§ 573). Every spherical polygon is assumed to be convex unless otherwise stated.

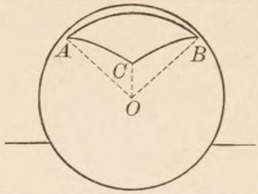
786. Def. A **diagonal of a spherical polygon** is an arc of a great circle connecting any two vertices which are not adjacent.

787. Def. A **spherical triangle** is a spherical polygon of three sides; like a plane triangle, it may be *right*, *obtuse*, or *acute*; *equilateral*, *isosceles*, or *scalene*.

788. Def. **Two spherical polygons are equal** if they can be applied, the one to the other, so as to coincide.

Proposition XI. Theorem.

789. *Each side of a spherical triangle is less than the sum of the other two sides.*



Let ABC be a spherical triangle, AB the longest side.

To prove that $AB < AC + BC$.

Proof. In the corresponding trihedral angle $O\text{-}ABC$,

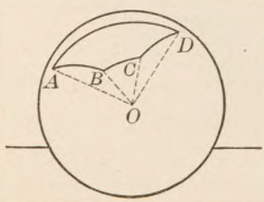
$\angle AOB$ is less than $\angle AOC + \angle BOC$. § 580

$\therefore AB < AC + BC$. § 783

Q. E. D.

Proposition XII. Theorem.

790. *The sum of the sides of a spherical polygon is less than 360°.*



Let ABCD be a spherical polygon.

To prove that $AB + BC + CD + DA < 360°$.

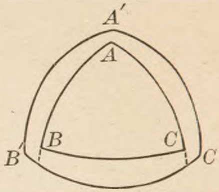
Proof. In the corresponding polyhedral angle $O\text{-}ABCD$, the sum of all the face angles is less than 360°. § 581

$\therefore AB + BC + CD + DA < 360°$. § 783

Q. E. D.

791. Def. If, from the vertices of a spherical triangle as poles, arcs of great circles are described, another spherical triangle is formed, called the **polar triangle** of the first.

Thus, if A is the pole of the arc of the great circle $B'C'$, B of $A'C'$, C of $A'B'$, $A'B'C'$ is the polar triangle of ABC.

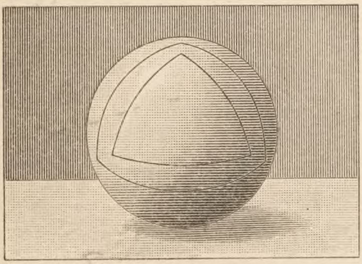


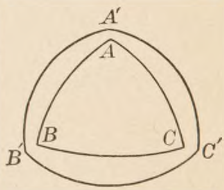
If, with A, B, C as poles, entire great circles are described, these circles divide the surface of the sphere into *eight* spherical triangles.

Of these eight triangles, that one is the polar of ABC whose vertex A', corresponding to A, lies on the same side of BC as the vertex A; and similarly for the other vertices.

Proposition XIII. Theorem.

792. *If $A'B'C'$ is the polar triangle of ABC, then, reciprocally, ABC is the polar triangle of $A'B'C'$.*





Let $A'B'C'$ be the polar triangle of ABC.

To prove that ABC is the polar triangle of $A'B'C'$.

Proof. A is the pole of $B'C'$, and C is the pole of $A'B'$. § 791

$\therefore$ B' is at a quadrant's distance from A and C. § 761

$\therefore$ B' is the pole of the arc AC. § 762

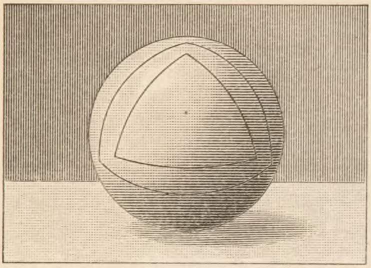
Similarly, A' is the pole of BC, and C' the pole of AB.

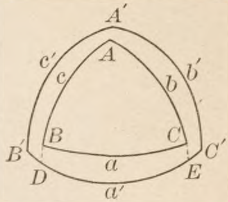
$\therefore$ ABC is the polar triangle of $A'B'C'$. § 791

Q. E. D.

Proposition XIV. Theorem.

793. *In two polar triangles each angle of the one is the supplement of the opposite side in the other.*





Let ABC, A′B′C′ be two polar triangles. Let the letter at the vertex of each angle denote its value in degrees, and the small letter the value of the opposite side in degrees.

To prove that $A + a' = 180°$, $B + b' = 180°$, $C + c' = 180°$;
$A' + a = 180°$, $B' + b = 180°$, $C' + c = 180°$.

Proof. Produce the arcs AB, AC until they meet $B'C'$ at the points D, E, respectively.

Now B' is the pole of AE. $\therefore B'E = 90°$. § 761

Also C' is the pole of AD. $\therefore C'D = 90°$.

Adding, $B'E + C'D = 180°$. Ax. 2

That is, $B'D + DE + C'D = 180°$.

Or $DE + B'C' = 180°$.

But DE is the measure of the $\angle A$, § 779

and $B'C' = a'$.

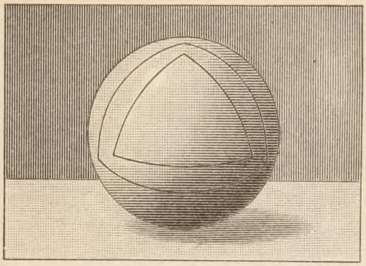
$\therefore A + a' = 180°$.

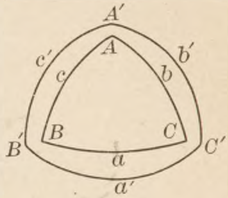
In a similar way all the other relations are proved. Q. E. D.

794. Def. Polar triangles are often called **supplemental triangles.**

PROPOSITION XV. THEOREM.

795. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*





Let ABC be a spherical triangle, and let A, B, C denote the values of its respective angles, and a′, b′, c′ the values of the opposite sides in the polar triangle A′B′C′.

To prove that $A + B + C > 180°$ *and* $< 540°$.

Proof. Since the $\triangle ABC, A'B'C'$, are polar $\triangle$,

$A + a' = 180°,\ B + b' = 180°,\ C + c' = 180°.$ § 793

$\therefore A + B + C + a' + b' + c' = 540°.$ Ax. 2

$\therefore A + B + C = 540° - (a' + b' + c').$

Now $a' + b' + c'$ is less than 360°. § 790

$\therefore A + B + C = 540° -$ some value less than 360°.

$\therefore A + B + C > 180°.$

Again $a' + b' + c'$ is greater than 0°.

$\therefore A + B + C < 540°.$ Q.E.D.

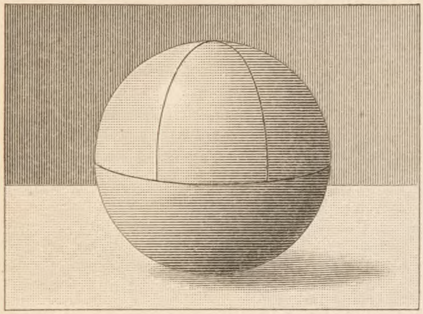
796. COR. *A spherical triangle may have two, or even three, right angles; and it may have two, or even three, obtuse angles.*

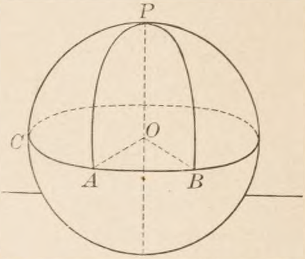
797. DEF. A spherical triangle having two right angles is called a **bi-rectangular** triangle; and a spherical triangle having three right angles is called a **tri-rectangular** triangle.

798. DEF. The **spherical excess** of a triangle is the difference between the sum of its angles and 180°.

PROPOSITION XVI. THEOREM.

799. *In a bi-rectangular spherical triangle the sides opposite the right angles are quadrants, and the side opposite the third angle measures that angle.*





Let PAB be a bi-rectangular triangle, with A, B right angles.

To prove that PA and PB are quadrants, and that the $\angle P$ *is measured by the arc AB.*

Proof. Since the $\angle s$ A and B are right angles, the planes of the arcs PA, PB are $\perp$ to the plane of the arc AB. § 780

$\therefore PA$ and PB must each pass through the pole of AB. § 754

$\therefore P$ is the pole of AB, and PA, PB are quadrants. § 761

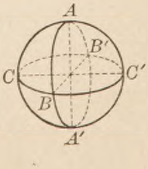
Also the $\angle P$ is measured by the arc AB. § 779

Q. E. D.

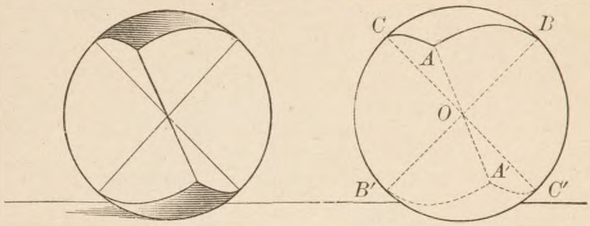
800. COR. 1. *If two sides of a spherical triangle are quadrants, the third side measures the opposite angle.*

801. COR. 2. *Each side of a tri-rectangular spherical triangle is a quadrant.*

802. COR. 3. *Three planes passed through the centre of a sphere, each perpendicular to the other two planes, divide the surface of the sphere into eight equal tri-rectangular triangles.*



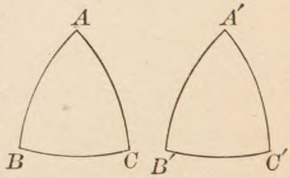
803. Def. If through the centre O of a sphere three diameters AA', BB', CC' are drawn, and the points A, B, C are joined by arcs of great circles, and also the points A', B', C', the two spherical triangles ABC and $A'B'C'$ are called **symmetrical** spherical triangles.



In the same way we may form two symmetrical polygons of any number of sides, and place each of them in any position we choose upon the surface of the sphere.

804. Two symmetrical triangles are mutually equilateral and equiangular; yet in general they cannot be made to coincide by superposition. If in the above figure the triangle ABC is made to slide on the surface of the sphere until the vertex A falls on A', it will be evident that the two triangles cannot be made to coincide and that the corresponding parts of the triangles occur in **reverse order.**

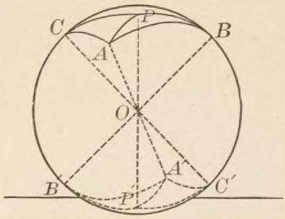
805. If, however, $AB = AC$, and $A'B' = A'C'$; that is, if the two symmetrical triangles are *isosceles*, then, because AB, AC, $A'B'$, $A'C'$ are all equal, and the angles A and A' are equal, being opposite dihedral angles (§ 803), the two triangles can be made to coincide. Therefore,



806. *If two symmetrical spherical triangles are isosceles, they are superposable and therefore equal.*

Proposition XVII. Theorem.

807. *Two symmetrical spherical triangles are equivalent.*



Let ABC, A′B′C′ be two symmetrical spherical triangles with their homologous vertices opposite each to each.

To prove that the triangles ABC, $A'B'C'$ are equivalent.

Proof. Let P be the pole of a small circle passing through the points A, B, C, and let POP' be a diameter.

Draw the great circle arcs PA, PB, PC, $P'A'$, $P'B'$, $P'C'$.

Then $\quad PA = PB = PC.$ § 758

Now $\quad P'A' = PA,\ P'B' = PB,\ P'C' = PC.$ § 804

$\therefore P'A' = P'B' = P'C'.$ Ax. 1

$\therefore$ the two symmetrical $\triangle$ PAC and $P'A'C'$ are isosceles.

$\therefore \triangle PAC = \triangle P'A'C'.$ § 806

Similarly, $\quad \triangle PAB = \triangle P'A'B',$

and $\quad \triangle PBC = \triangle P'B'C'.$

Now $\quad \triangle ABC \approx \triangle PAC + \triangle PAB + \triangle PBC,$ Ax. 9

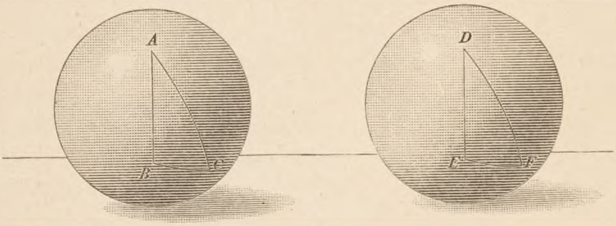
and $\quad \triangle A'B'C' \approx \triangle P'A'C' + \triangle P'A'B' + \triangle P'B'C'.$

$\therefore \triangle ABC \approx \triangle A'B'C'.$ Q. E. D.

If the pole P should fall without the $\triangle ABC$, then P' would fall without $\triangle A'B'C'$, and each triangle would be equivalent to the sum of two symmetrical isosceles triangles diminished by the third; so that the result would be the same as before.

Proposition XVIII. Theorem.

808. *Two triangles on the same sphere or equal spheres are equal, if two sides and the included angle, or two angles and the included side, of the one are respectively equal to the corresponding parts of the other and arranged in the same order.*

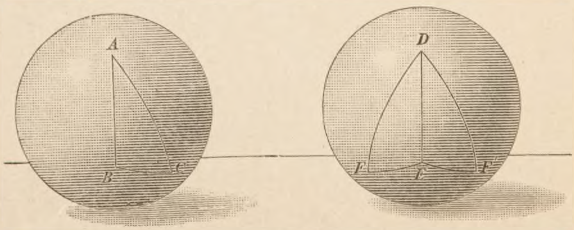


Proof. By superposition, as in plane △. §§ 143, 139

Q. E. D.

Proposition XIX. Theorem.

809. *Two triangles on the same sphere or equal spheres are symmetrical, if two sides and the included angle, or two angles and the included side, of the one are equal, respectively, to the corresponding parts of the other and arranged in the reverse order.*



Proof. Construct the △ DEF' symmetrical with respect to the △ DEF upon the same sphere.

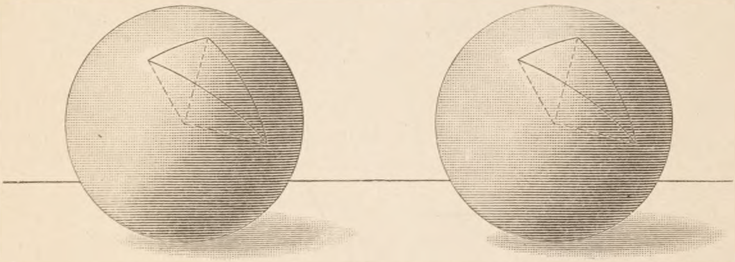
Then $\triangle ABC$ can be superposed upon the $\triangle DEF'$, so that they will coincide as in the corresponding case of plane $\triangle$.

But $\triangle DEF'$ and DEF are symmetrical by construction.

$\therefore \triangle ABC$, which coincides with $\triangle DEF'$, is symmetrical with respect to $\triangle DEF$. Q. E. D.

Proposition XX. Theorem.

810. *Two mutually equilateral triangles on the same sphere or equal spheres are mutually equiangular, and are equal or symmetrical.*



Proof. The face $\angle$ of the corresponding trihedral $\angle$ at the centre of the sphere are equal respectively. § 237

Therefore, the corresponding dihedral $\angle$ are equal. § 583

Hence, the $\angle$ of the spherical $\triangle$ are respectively equal.

Therefore, the $\triangle$ are equal or symmetrical, according as their equal sides are arranged in the same or reverse order.

Q. E. D.

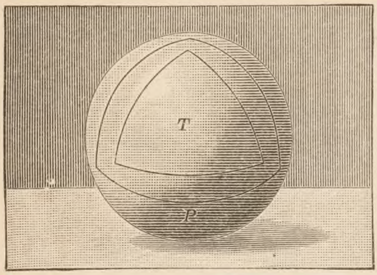
Ex. 737. The radius of a sphere is 4 inches. From any point on the surface as a pole a circle is described upon the sphere with an opening of the compasses equal to 3 inches. Find the area of this circle. = 24,28

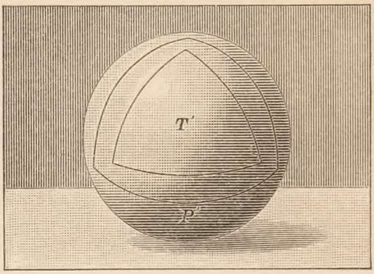
Ex. 738. The edge of a regular tetrahedron is a. Find the radii R, R' of the inscribed and circumscribed spheres.

Ex. 739. Find the diameter of the section of a sphere 10 inches in diameter made by a plane 3 inches from the centre. 4 inches

Proposition XXI. Theorem.

811. *Two mutually equiangular triangles on the same sphere or equal spheres are mutually equilateral, and are either equal or symmetrical.*





Let the spherical triangles T and T′ be mutually equiangular.

To prove that T and T' are mutually equilateral, and equal or symmetrical.

Proof. Let the $\triangle$ P be the polar $\triangle$ of T, and P' of T'.

By hypothesis, the $\triangle$ T and T' are mutually equiangular.

$\therefore$ the polar $\triangle$ P and P' are mutually equilateral. § 793

$\therefore$ the polar $\triangle$ P and P' are mutually equiangular. § 810

But the $\triangle$ T and T' are the polar $\triangle$ of P and P'. § 792

$\therefore$ the $\triangle$ T and T' are mutually equilateral. § 793

Hence, the $\triangle$ T and T' are equal or symmetrical. § 810

Q.E.D.

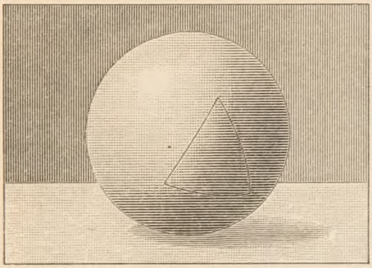
Note. The statement that mutually equiangular spherical triangles are mutually equilateral, and equal or symmetrical, is true only when they are on the same sphere, or equal spheres. But when the spheres are unequal, the spherical triangles are unequal; and the ratio of their homologous sides is equal to the ratio of the radii of the spheres. § 465

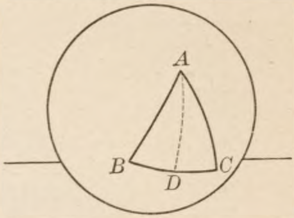
Ex. 740. At a given point in a given arc of a great circle, to construct a spherical angle equal to a given spherical angle.

Ex. 741. To inscribe a circle in a given spherical triangle.

PROPOSITION XXII. THEOREM.

812. *In an isosceles spherical triangle, the angles opposite the equal sides are equal.*





In the spherical triangle ABC, let AB equal AC.

To prove that $\angle B = \angle C$.

Proof. Draw the arc AD of a great circle, from the vertex A to the middle of the base BC.

Then $\triangle$ ABD and ACD are mutually equilateral.

$\therefore$ $\triangle$ ABD and ACD are mutually equiangular. § 810

$\therefore \angle B = \angle C$. Q.E.D.

813. COR. *The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle of the base bisects the vertical angle, is perpendicular to the base, and divides the triangle into two symmetrical triangles.*

Ex. 742. To circumscribe a circle about a given spherical triangle.

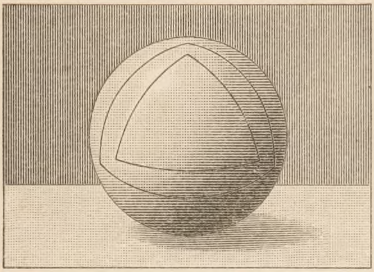
Ex. 743. Given a spherical triangle whose sides are 60°, 80°, and 100°. Find the angles of its polar triangle.

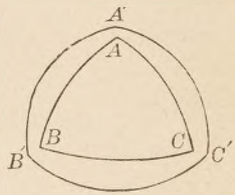
Ex. 744. Given a spherical triangle whose angles are 70°, 75°, and 95°. Find the sides of its polar triangle.

Ex. 745. Find the ratio of two homologous sides of two mutually equiangular triangles on spheres whose radii are 12 inches and 20 inches.

PROPOSITION XXIII. THEOREM.

814. *If two angles of a spherical triangle are equal, the sides opposite these angles are equal and the triangle is isosceles.*





In the spherical triangle ABC, let the angle B equal the angle C.

To prove that $AC = AB$.

Proof. Let the $\triangle A'B'C'$ be the polar $\triangle$ of the $\triangle ABC$.

Now $\angle B = \angle C$. Hyp.

$\therefore A'C' = A'B'$. § 793

$\therefore \angle B' = \angle C'$. § 812

$\therefore AC = AB$. § 793

Q.E.D.

Ex. 746. To bisect a spherical angle.

Ex. 747. To construct a spherical triangle, having given two sides and the included angle.

Ex. 748. To construct a spherical triangle, having given two angles and the included side.

Ex. 749. To construct a spherical triangle, having given the three sides.

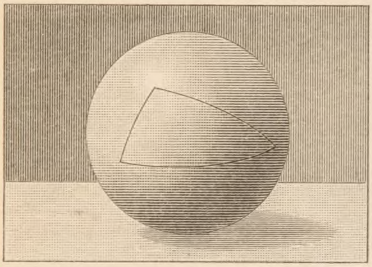
Ex. 750. To construct a spherical triangle, having given the three angles.

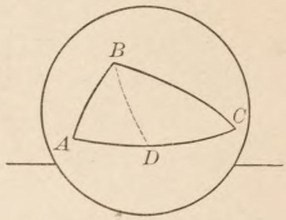
Ex. 751. To pass a plane tangent to a sphere at a given point on the surface of the sphere.

Ex. 752. To pass a plane tangent to a sphere through a given straight line without the sphere.

Proposition XXIV. Theorem.

815. *If two angles of a spherical triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle;* CONVERSELY, *if two sides are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.*





1. **In the triangle ABC, let the angle ABC be greater than the angle ACB.**

To prove that $AC > AB$.

Proof. Draw the arc BD of a great circle, making $\angle CBD$ equal $\angle ACB$. Then $DC = DB$. § 814

Now $AD + DB > AB$. § 789

$\therefore AD + DC > AB$, or $AC > AB$.

2. **Let AC be greater than AB.**

To prove that the $\angle ABC$ *is greater than the* $\angle ACB$.

Proof. The $\angle ABC$ must be equal to, less than, or greater than the $\angle ACB$.

If $\angle ABC = \angle C$, then $AC = AB$; § 814

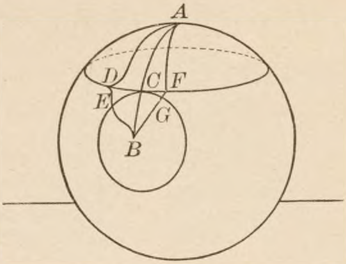
and if $\angle ABC$ is less than $\angle C$, then $AC < AB$. (1)

But both of these conclusions are contrary to the hypothesis.

$\therefore \angle ABC$ is greater than $\angle C$. Q. E. D.

Proposition XXV. Theorem.

816. *The shortest line that can be drawn on the surface of a sphere between two points is the arc of a great circle, joining the two points not greater than a semicircumference.*



Let AB be the arc of a great circle, not greater than a semicircumference, joining the points A and B.

To prove that AB is the shortest line that can be drawn on the surface joining A and B.

Proof. Let C be any point in AB.

With A and B as poles and AC and BC as polar distances, describe two arcs DCF and ECG.

The arcs DCF and ECG have only the point C in common. For if F is any other point in DCF, and if arcs of great circles AF and BF are drawn, then

$$AF = AC. \qquad \S\ 758$$

But $$AF + BF > AC + BC. \qquad \S\ 789$$

Take away AF from the left member of the inequality, and its equal AC from the right member.

Then $$BF > BC. \qquad \text{Ax. 5}$$

Therefore, $BF > BG$, the equal of BC.

Hence, F lies without the circumference whose pole is B, and the arcs DCF and ECG have only the point C in common.

Now let $ADEB$ be any line from A to B on the surface of the sphere, which does not pass through C.

This line will cut the arcs DCF and ECG in separate points D and E, and if we revolve the line AD about A as a fixed point until D coincides with C we shall have a line from A to C equal to the line AD.

In like manner, we can draw a line from B to C equal to the line BE.

Therefore, a line can be drawn from A to B through C that is equal to the sum of the lines AD and BE, and hence less than the line $ADEB$ by the line DE.

Therefore, no line which does not pass through C can be the shortest line from A to B.

Therefore, the shortest line from A to B passes through C.

But C is *any* point in the arc AB.

Therefore, the shortest line from A to B passes through *every* point of the arc AB, and consequently coincides with the arc AB.

Therefore, the shortest line from A to B is the great circle arc AB. Q. E. D.

Ex. 753. The three medians of a spherical triangle meet in a point.

Ex. 754. To construct a spherical surface with a given radius that passes through three given points.

Ex. 755. To construct a spherical surface with a given radius that passes through two given points and is tangent to a given plane.

Ex. 756. To construct a spherical surface with a given radius that passes through two given points and is tangent to a given sphere.

Ex. 757. All arcs of great circles drawn through a pole of a given great circle are perpendicular to the circumference of the great circle.

Ex. 758. The smallest circle whose plane passes through a given point within a sphere is the one whose plane is perpendicular to the radius through the given point.

MEASUREMENT OF SPHERICAL SURFACES.

817. Def. A **zone** is a portion of the surface of a sphere included between two parallel planes.

The circumferences of the sections made by the planes are called the **bases of the zone**, and the distance between the planes is the **altitude of the zone.**

818. Def. A **zone of one base** is a zone one of whose bounding planes is tangent to the sphere.

If a great circle $PADQ$ (Fig. 1) is revolved about its diameter PQ, the arc AD will generate a zone, the points A and D will generate its bases, and CF will be its altitude.

The arc PA will generate a zone of one base.

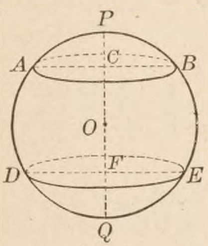


Fig. 1.

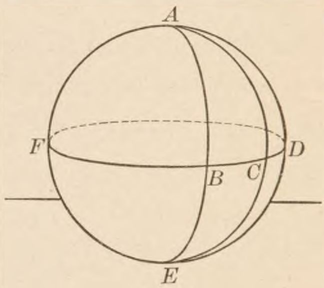


Fig. 2.

819. Def. A **lune** is a portion of the surface of a sphere bounded by two semicircumferences of great circles.

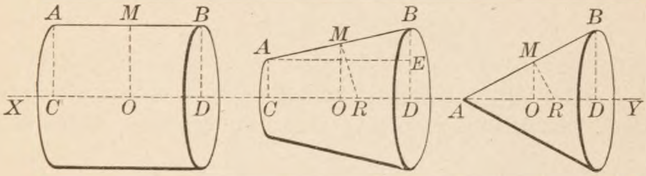
820. Def. The **angle** of a lune is the angle between the semicircumferences which form its boundaries.

Thus (Fig. 2), $ABECA$ is a lune, BAC is its angle.

821. Def. It is convenient to divide each of the eight equal tri-rectangular triangles of which the surface of a sphere is composed (§ 802) into 90 equal parts, and to call each of these parts a **spherical degree.** The surface of every sphere, therefore, contains 720 spherical degrees.

Proposition XXVI. Theorem.

822. *The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line on the axis by the circumference whose radius is a perpendicular erected at the middle point of the line and terminated by the axis.*



Let XY be the axis, AB the revolving line, M its middle point, CD its projection on XY, MO perpendicular to XY, and MR to AB.

To prove that the area $AB = CD \times 2\,\pi MR$.

Proof. 1. If AB is $\parallel$ to XY, $CD = AB$, MR coincides with MO, and the area AB is the surface of a right cylinder. § 697

2. If AB is not $\parallel$ to XY, and does not cut XY, the area AB is the surface of the frustum of a cone of revolution.

$\therefore$ the area $AB = AB \times 2\,\pi MO$. § 728

Draw $AE \parallel$ to XY.

The $\triangle$ ABE and MOR are similar. § 359

$\therefore MO : AE = MR : AB$. § 351

$\therefore AB \times MO = AE \times MR$. § 327

Or $AB \times MO = CD \times MR$.

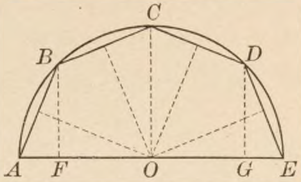
Substituting, the area $AB = CD \times 2\,\pi MR$.

3. If A lies in the axis XY, the reasoning still holds, but AE and CD coincide, and the truth follows from § 722.

Q. E. D.

Proposition XXVII. Theorem.

823. *The area of the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*



Let S denote the surface, R the radius, of a sphere generated by the semicircle ABCDE revolving about the diameter AE as an axis.

To prove that $S = AE \times 2\pi R$.

Proof. Inscribe in the semicircle half of a regular polygon having an *even* number of sides, as $ABCDE$.

From the centre draw $\perp$s to the chords AB, BC, etc.

These $\perp$s bisect the chords (§ 245) and are equal. § 249

Let a denote the length of each of these $\perp$s.

From B, C, and D drop the $\perp$s BF, CO, and DG to AE.

Then the area $AB = AF \times 2\pi a$, § 822

the area $BC = FO \times 2\pi a$, etc.

$\therefore$ the area $ABCDE = AE \times 2\pi a$.

Denote the area of the surface described by the semi-polygon by S', and let the number of sides of the semi-polygon be indefinitely increased.

Then S' approaches S as a limit,

and a approaches R as a limit. § 449

$\therefore AE \times 2\pi a$ approaches $AE \times 2\pi R$ as a limit. § 279

But $S' = AE \times 2\pi a$, always. § 822

$\therefore S = AE \times 2\pi R$. § 284

Q. E. D.

824. Cor. 1. *The surface of a sphere is equivalent to four great circles; that is, to* $4\pi R^2$.

For πR^2 is equal to the area of a great circle, § 463
and $4\pi R^2$ is equal to $2R \times 2\pi R$, the area of the surface of a sphere. § 823

825. Cor. 2. *The areas of the surfaces of two spheres are as the squares of their radii, or as the squares of their diameters.*

Let R and R' denote the radii, D and D' the diameters, and S and S' the areas of the surfaces of two spheres.

Then $$\frac{S}{S'} = \frac{4\pi R^2}{4\pi R'^2} = \frac{R^2}{R'^2} = \frac{(\frac{1}{2}D)^2}{(\frac{1}{2}D')^2} = \frac{D^2}{D'^2}.$$

826. Cor. 3. *The area of a zone is equal to the product of its altitude by the circumference of a great circle.*

If we apply the reasoning of § 823 to the zone generated by the revolution of the arc BCD, we obtain

$$\text{the area of zone } BCD = FG \times 2\pi R,$$

where FG is the altitude of the zone and $2\pi R$ the circumference of a great circle.

827. Cor. 4. *Zones on the same sphere or equal spheres are to each other as their altitudes.*

828. Cor. 5. *A zone of one base is equivalent to a circle whose radius is the chord of the generating arc.*

The arc AB generates a zone of one base;

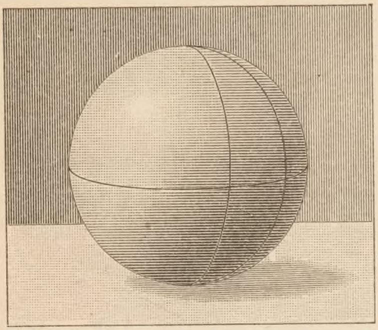
and zone $AB = AF \times 2\pi R = \pi AF \times AE$.

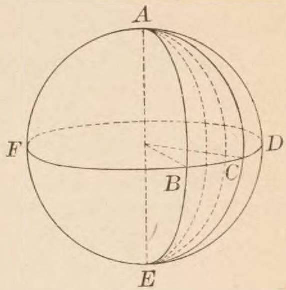
But $AF \times AE = \overline{AB}^2$. § 370

$\therefore$ the zone $AB = \pi\overline{AB}^2$.

Proposition XXVIII. Theorem.

829. *The area of a lune is to the area of the surface of the sphere as the number of degrees in the angle of the lune is to 360.*





Let ABEC be a lune, BCDF the great circle whose pole is A; also let A denote the number of degrees in the angle of the lune, L the area of the lune, and S the area of the surface of the sphere.

To prove that $L : S = A : 360.$

Proof. The arc BC measures the $\angle A$ of the lune. § 779

Hence, arc BC : circumference $BCDF = A : 360.$

If BC and $BCDF$ are commensurable, let their common measure be contained m times in BC, and n times in $BCDF$.

Then arc BC : circumference $BCDF = m : n.$

$\therefore A : 360 = m : n.$ § 288

Pass arcs of great Ⓢ through the diameter AE and all the points of the division of $BCDF$. These arcs will divide the entire surface into n equal lunes, of which the lune $ABEC$ will contain m.

$\therefore L : S = m : n.$

$\therefore L : S = A : 360.$ Ax. 1

If BC and $BCDF$ are incommensurable, the theorem can be proved by the method of limits as in § 549. Q.E.D.

830. Cor. 1. *The number of spherical degrees in a lune is equal to twice the number of angle degrees in the angle of the lune.*

If L and S are expressed in spherical degrees, § 821

then $L : 720 = A : 360.$ § 829

Therefore, $L = 2A.$

831. Cor. 2. *The area of a lune is equal to one ninetieth of the area of a great circle multiplied by the number of degrees in the angle of the lune.*

For $L : 4\pi R^2 = A : 360.$ § 829

Therefore, $L = \dfrac{\pi R^2 A}{90}.$

832. Cor. 3. *Two lunes on the same sphere or equal spheres have the same ratio as their angles.*

For $L : L' = \dfrac{\pi R^2 A}{90} : \dfrac{\pi R^2 A'}{90}.$ § 831

That is, $L : L' = A : A'.$

833. Cor. 4. *Two lunes which have equal angles, but are situated on unequal spheres, have the same ratio as the squares of the radii of the spheres on which they are situated.*

For $L : L' = \dfrac{\pi R^2 A}{90} : \dfrac{\pi R'^2 A}{90}.$ § 831

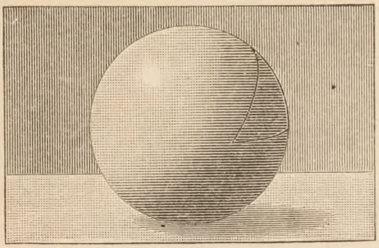
That is, $L : L' = R^2 : R'^2.$

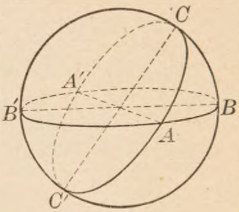
Ex. 759. Given the radius of a sphere 10 inches. Find the area of a lune whose angle is 30°. = 104,72

Ex. 760. Given the diameter of a sphere 16 inches. Find the area of a lune whose angle is 75°. = 167,43

Proposition XXIX. Theorem.

834. *The area of a spherical triangle, expressed in spherical degrees, is numerically equal to the spherical excess of the triangle.*





Let A, B, C denote the values of the angles of the spherical triangle ABC, and E the spherical excess.

To prove that the number of spherical degrees in $\triangle ABC = E$.

Proof. Produce the sides of the $\triangle ABC$ to complete circles.

These circles divide the surface of the sphere into eight spherical triangles, of which any four having a common vertex, as A, form the surface of a hemisphere.

The $\triangle s$ $A'BC$, $AB'C'$ are symmetrical and equivalent. § 807

And $\quad \triangle ABC + \triangle A'BC \approx$ lune $ABA'C$.

Put the $\triangle AB'C'$ for its equivalent, the $\triangle A'BC$.

Then $\quad \triangle ABC + \triangle AB'C' \approx$ lune $ABA'C$.

Also $\quad \triangle ABC + \triangle AB'C \approx$ lune $BAB'C$.

And $\quad \triangle ABC + \triangle ABC' \approx$ lune $CAC'B$.

Add and observe that in spherical degrees

$$\triangle ABC + \triangle AB'C' + \triangle AB'C + \triangle ABC' = 360, \quad \text{§ 821}$$

and $\quad ABA'C + BAB'C + CAC'B = 2(A + B + C)$. § 830

Then $\quad 2\triangle ABC + 360 = 2(A + B + C)$.

$$\therefore \triangle ABC = A + B + C - 180 = E. \quad \text{Q.E.D.}$$

835. Cor. 1. *The area of a spherical triangle is to the area of the surface of the sphere as the number which expresses its spherical excess is to* 720.

For the number of spherical degrees in a spherical $\triangle ABC$ is equal to E (§ 834), and the number of spherical degrees in S, the surface of the sphere, is equal to 720. § 821

$$\therefore \triangle ABC : S = E : 720.$$

836. Cor. 2. *The area of a spherical triangle is equal to the area of a great circle multiplied by the number of degrees in* E *divided by one hundred eighty.*

For $\triangle ABC : S = E : 720.$ § 835

But $S = 4\pi R^2.$ § 824

$$\therefore \triangle ABC = \frac{4\pi R^2 E}{720} = \frac{\pi R^2 E}{180}.$$

Ex. 761. What part of the surface of a sphere is a triangle whose angles are 120°, 100°, and 95°? What is its area in square inches, if the radius of the sphere is 6 inches?

Ex. 762. Find the area of a spherical triangle whose angles are 100°, 120°, 140°, if the diameter of the sphere is 16 inches.

Ex. 763. If the radii of two spheres are 6 inches and 4 inches respectively, and the distance between their centres is 5 inches, what is the area of the circle of intersection of these spheres?

Ex. 764. Find the radius of the circle determined in a sphere of 5 inches diameter by a plane 1 inch from the centre.

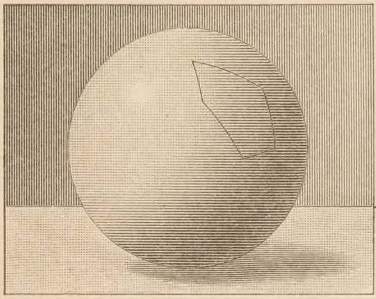
Ex. 765. If the radii of two concentric spheres are R and R', and if a plane is drawn tangent to the interior sphere, what is the area of the section made in the other sphere?

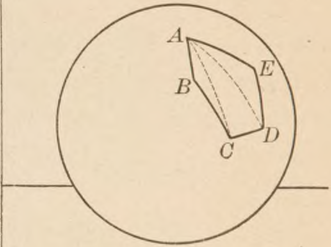
Ex. 766. Two points A and B are 8 inches apart. Find the locus in space of a point 5 inches from A and 7 inches from B.

Ex. 767. The radii of two parallel sections of the same sphere are a and b respectively, and the distance between these sections is d. Find the radius of the sphere.

Proposition XXX. Theorem.

837. *If T denotes the number which expresses the sum of the angles of a spherical polygon of n sides, the area of the polygon expressed in spherical degrees is numerically equal to $T - (n - 2)\,180$.*





Let ABCDE be a polygon of n sides.

To prove that the area of $ABCDE$ expressed in spherical degrees is numerically equal to

$$T - (n - 2)\,180.$$

Proof. Divide the polygon into spherical triangles by drawing diagonals from any vertex, as A.

These diagonals divide the polygon into $n - 2$ spherical $\triangle$.

The area of each triangle in spherical degrees is numerically equal to the sum of its angles minus 180. § 834

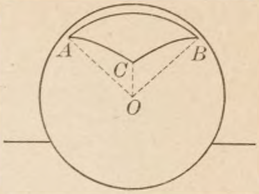
Hence, the sum of the areas of all the $n - 2$ triangles expressed in spherical degrees is numerically equal to the sum of all their angles minus $(n - 2)\,180$.

Now the sum of the areas of the triangles is the area of the polygon, and the sum of the angles of the triangles is the sum of the angles of the polygon.

Therefore, the area of the polygon expressed in spherical degrees is numerically equal to $T - (n - 2)\,180$. Q.E.D.

SPHERICAL VOLUMES.

838. Def. A **spherical pyramid** is the portion of a sphere bounded by a spherical polygon and the planes of its sides.

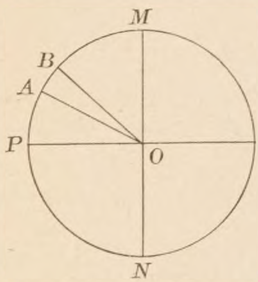


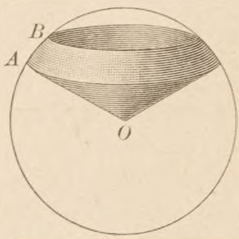
The centre of the sphere is the **vertex** of the pyramid, and the spherical polygon is the **base** of the pyramid.

Thus, $O\text{-}ABC$ is a spherical pyramid.

839. Def. A **spherical sector** is the portion of a sphere generated by the revolution of a circular sector about any diameter of the circle of which the sector is a part.

The **base** of a spherical sector is the zone generated by the arc of the circular sector.





840. Def. A **spherical segment** is a portion of a sphere contained between two parallel planes.

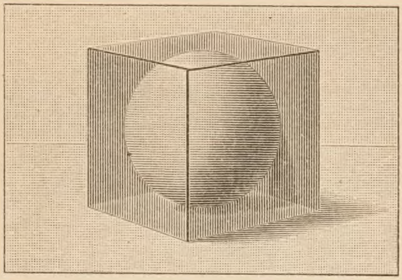
841. Def. The **bases** of a spherical segment are the sections made by the parallel planes, and the **altitude** of a spherical segment is the perpendicular distance between the bases.

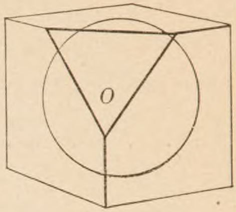
842. Def. If one of the parallel planes is tangent to the sphere, the segment is called a **segment of one base**.

843. Def. A **spherical wedge** is a portion of a sphere bounded by a lune and two great semicircles.

Proposition XXXI. Theorem.

844. *The volume of a sphere is equal to the product of the area of its surface by one third of its radius.*





Let R be the radius of a sphere whose centre is O, S its surface, and V its volume.

To prove that $V = S \times \frac{1}{3} R.$

Proof. Conceive a cube to be circumscribed about the sphere. Its volume is greater than that of the sphere, because it contains the sphere.

From O, the centre of the sphere, conceive lines to be drawn to the vertices of the cube.

These lines are the edges of six quadrangular pyramids, whose bases are the faces of the cube, and whose common altitude is the radius of the sphere.

The volume of each pyramid is equal to the product of its base by $\frac{1}{3}$ its altitude. Hence, the volume of the six pyramids, that is, the volume of the circumscribed cube, is equal to the area of the surface of the cube multiplied by $\frac{1}{3} R$.

Now conceive planes drawn tangent to the sphere, at the points where the edges of the pyramids cut its surface. We then have a circumscribed solid whose volume is nearer that of the sphere than is the volume of the circumscribed cube, because each tangent plane cuts away a portion of the cube.

From O conceive lines to be drawn to each of the polyhedral angles of the solid thus formed. These lines form the edges of a series of pyramids, whose bases are together equal to the surface of the solid, and whose common altitude is the radius of the sphere; and the volume of each pyramid thus formed is equal to the product of its base by $\frac{1}{3}$ its altitude.

Hence, the sum of the volumes of these pyramids, that is, the volume of this new solid, is again equal to the area of its surface multiplied by $\frac{1}{3}R$.

If we denote the area of the surface of this polyhedron by S', and the volume of the polyhedron by V',

$$V' = S' \times \tfrac{1}{3}R.$$

If we continue to draw tangent planes to the sphere, we continue to diminish the circumscribed solid, since each new plane cuts off a corner of the polyhedron.

By continuing this process indefinitely, we can make the difference between the volumes of the circumscribed solid and sphere less than any assigned quantity, however small, but we cannot make it zero; and the difference between the areas of the surfaces of the circumscribed solid and sphere less than any assigned quantity, however small, but we cannot make it zero.

Hence, V is the limit of V', and S is the limit of S'. § 275

But $V' = S' \times \frac{1}{3}R$, always.

$\therefore V = S \times \frac{1}{3}R.$ § 284

Q. E. D.

845. Cor. 1. *The volume of a sphere is equal to* $4\pi R^2 \times \frac{1}{3}R$*; that is,* $\frac{4}{3}\pi R^3$*, or* $\frac{1}{6}\pi D^3$. ($D =$ diameter.)

846. Cor. 2. *The volumes of two spheres are to each other as the cubes of their radii.*

For $V : V' = \frac{4}{3}\pi R^3 : \frac{4}{3}\pi R'^3 = R^3 : R'^3.$

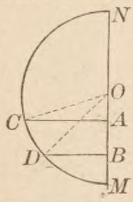
847. Cor. 3. *The volume of a spherical pyramid is equal to the product of its base by one third of the radius of the sphere.*

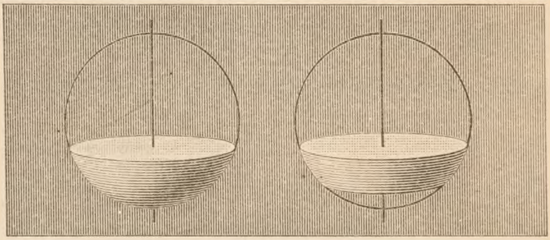
848. Cor. 4. *The volume of a spherical sector is equal to one third the product of the zone which forms its base by the radius of the sphere.*

If R denotes the radius of the sphere, C the circumference of a great circle, H the altitude of the zone, Z the surface of the zone, and V the volume of the sector; then, $C = 2\,\pi R$ (§ 458), $Z = 2\,\pi R \times H$ (§ 826), and $V = 2\,\pi RH \times \frac{1}{3}R = \frac{2}{3}\,\pi R^2 H.$

Proposition XXXII. Problem.

849. *To find the volume of a spherical segment.*





Let AC and BD be two semi-chords perpendicular to the diameter MN of the semicircle NCDM. Let OM be equal to R, AM to a, BM to b, AB to h, AC to r, BD to r′.

Case 1. *To find the volume of the segment of one base generated by the circular semi-segment ACM, as the semicircle revolves about MN as an axis.*

The sector generated by $OCM = \frac{2}{3}\,\pi R^2 a.$ § 848

The cone generated by $OCA = \frac{1}{3}\,\pi r^2 (R - a).$ § 724

Hence, segment $ACM = \frac{2}{3}\,\pi R^2 a - \frac{1}{3}\,\pi r^2 (R - a)$

$$= \frac{\pi}{3}(2\,R^2 a - Rr^2 + ar^2).$$

Now $r^2 = AM \times AN = a(2\,R - a).$ § 370

$$\therefore \text{ the segment } ACM = \pi a^2\left(R - \frac{a}{3}\right). \qquad (1)$$

If from the relation $r^2 = a(2R - a)$ we find the value of R, and substitute it in (1), we obtain the volume in terms of the altitude and the radius of the base.

$$\text{The segment } ACM = \tfrac{1}{2}\pi r^2 a + \tfrac{1}{6}\pi a^3. \qquad (2)$$

CASE 2. *To find the volume of the segment of two bases generated by the circular semi-segment ABDC, as the semicircle revolves about NM as an axis.*

Since the volume is obviously the difference of the volumes of the segments of one base generated by the circular semi-segments ACM and BDM, therefore, by formula (1),

$$\text{segment } ABDC = \pi a^2\left(R - \frac{a}{3}\right) - \pi b^2\left(R - \frac{b}{3}\right)$$

$$= \pi R(a^2 - b^2) - \frac{\pi}{3}(a^3 - b^3) \qquad (3)$$

(put h for $a - b$)
$$= \pi Rh(a + b) - \frac{\pi h}{3}(a^2 + ab + b^2)$$

$$= \pi h\left[(Ra + Rb) - \tfrac{1}{3}(a^2 + ab + b^2)\right].$$

Now $a - b = h. \quad \therefore a^2 - 2ab + b^2 = h^2.$

Add $3ab$ to each side, $a^2 + ab + b^2 = h^2 + 3ab.$

Since $(2R - a)a = r^2$, and $(2R - b)b = r'^2$, § 370

$$Ra + Rb = \frac{r^2 + r'^2}{2} + \frac{a^2 + b^2}{2}.$$

$$\therefore \text{the segment } ABDC = \pi h\left[\frac{r^2 + r'^2}{2} + \frac{a^2 + b^2}{2} - \frac{h^2}{3} - ab\right]$$

$$= \pi h\left[\frac{r^2 + r'^2}{2} + \frac{h^2}{2} + ab - \frac{h^2}{3} - ab\right]$$

$$= \frac{h}{2}(\pi r^2 + \pi r'^2) + \frac{\pi h^3}{6}.$$

Q. E. F.

NUMERICAL EXERCISES.

Ex. 768. Find the surface of a sphere, if the diameter is (i) 10 inches; (ii) 1 foot 9 inches; (iii) 2 feet 4 inches; (iv) 7 feet; (v) 10.5 feet.

Ex. 769. Find the diameter of a sphere if the surface is (i) 616 square inches; (ii) $38\frac{1}{2}$ square feet; (iii) 9856 square feet.

Ex. 770. The circumference of a dome in the shape of a hemisphere is 66 feet. How many square feet of lead are required to cover it?

Ex. 771. If the ball on the top of St. Paul's Cathedral in London is 6 feet in diameter, what would it cost to gild it at 7 cents per square inch?

Ex. 772. What is the numerical value of the radius of a sphere, if its surface has the same numerical value as the circumference of a great circle?

Ex. 773. Find the surface of a lune, if its angle is 30°, and the total surface of the sphere is 4 square feet.

Ex. 774. What fractional part of the whole surface of a sphere is a spherical triangle whose angles are 43° 27′, 81° 57′, and 114° 36′?

Ex. 775. The angles of a spherical triangle are 60°, 70°, and 80°. The radius of the sphere is 14 feet. Find the area of the triangle.

Ex. 776. The sides of a spherical triangle are 80°, 74°, and 128°. The radius of the sphere is 14 feet. Find the area of the polar triangle in square feet.

Ex. 777. Find the area of a spherical polygon on a sphere whose radius is $10\frac{1}{2}$ feet, if its angles are 100°, 120°, 140°, and 160°.

Ex. 778. The planes of the faces of a quadrangular spherical pyramid make with each other angles of 80°, 100°, 120°, and 150°; and the length of a lateral edge of the pyramid is 42 feet. Find the area of its base in square feet.

Ex. 779. The planes of the faces of a triangular spherical pyramid make with each other angles of 60°, 80°, and 100°, and the area of the base of the pyramid is 4π square feet. Find the radius of the sphere.

Ex. 780. The diameter of a sphere is 21 feet. Find the curved surface of a segment whose height is 5 feet.

Ex. 781. In a sphere whose radius is R, find the height of a zone whose area is equal to that of a great circle.

Ex. 782. What is the area of a zone of one base whose height is h, and the radius of the base r? What would be the area if the height were twice as great?

Ex. 783. The altitude of the torrid zone is 3200 miles. Find its area, assuming the earth to be a sphere with a radius of 4000 miles.

Ex. 784. A plane divides the surface of a sphere of radius R into two zones, such that the surface of the greater is the mean proportional between the entire surface and the surface of the smaller. Find the distance of the plane from the centre of the sphere. $= -2R + R\sqrt{5}$

Ex. 785. If a sphere of radius R is cut by two parallel planes equally distant from the centre, so that the area of the zone comprised between the planes is equal to the sum of the areas of its bases, find the distance of either plane from the centre. $= h = R(\sqrt{8}-2)$

Ex. 786. Find the area of the zone generated by an arc of 30°, of which the radius is r, and which turns around a diameter passing through one of its extremities. $= \pi r^2(2-\sqrt{3})$

Ex. 787. Find the area of the zone of a sphere of radius R, illuminated by a lamp placed at the distance h from the sphere. $\frac{\pi R^2 h}{R+h}$

Ex. 788. How much of the earth's surface would a man see if he were raised to the height of the radius above it? $= \pi R^2 = \pi 4000$

Ex. 789. To what height must a man be raised above the earth in order that he may see one sixth of its surface? $h = \frac{R}{2}$

Ex. 790. The square on the diameter of a sphere and the square on an edge of the inscribed cube are as 3 : 1.

Ex. 791. Find the volume of a sphere, if the diameter is (i) 13 inches; (ii) 3 feet 6 inches; (iii) 10 feet 6 inches; (iv) 14.7 feet.

Ex. 792. Find the diameter of a sphere, if the volume is (i) 75 cubic feet 1377 cubic inches; (ii) 179 cubic feet 1152 cubic inches; (iii) 1047.816 cubic feet; (iv) 38.808 cubic yards.

Ex. 793. Find the volume of a sphere whose circumference is 45 feet.

Ex. 794. Find the volume V of a sphere in terms of the circumference C of a great circle. $V = \frac{C^3}{6\pi^2}$

Ex. 795. Find the radius R of a sphere, having given the volume V. $R = \sqrt[3]{\frac{3V}{4\pi}}$

Ex. 796. Find the radius R of a sphere, if its circumference and its volume have the same numerical value.

Ex. 797. The volume of a sphere is to the volume of the circumscribed cube as π is to 6.

Ex. 798. An iron ball 4 inches in diameter weighs 9 pounds. Find the weight of an iron shell 2 inches thick, whose external diameter is 20 inches.

Ex. 799. The radius of a sphere is 7 feet. What is the volume of a wedge whose angle is 36°?

Ex. 800. What is the angle of a spherical wedge, if its volume is one cubic foot, and the volume of the entire sphere is 6 cubic feet? 60°

Ex. 801. Find the volume of a spherical sector, if the area of the zone of its base is 3 square feet, and the radius of the sphere is 1 foot.

Ex. 802. The radius of the base of a segment of a sphere is 16 inches, and the radius of the sphere is 20 inches. Find the volume of the segment.

Ex. 803. The inside of a wash-basin is in the shape of the segment of a sphere; the distance across the top is 16 inches, and its greatest depth is 6 inches. Find how many pints of water it will hold, reckoning $7\frac{1}{2}$ gallons to the cubic foot.

Ex. 804. What is the height of a zone, if its area is S, and the volume of the sphere to which it belongs is V?

Ex. 805. The radii of the bases of a spherical segment are 6 feet and 8 feet, and its height is 3 feet. Find its volume.

Ex. 806. Find the volume of a triangular spherical pyramid, if the angles of the spherical triangle which forms its base are each 100°, and the radius of the sphere is 7 feet.

Ex. 807. The circumference of a sphere is 28π feet. Find the volume of that part of the sphere included by the faces of a trihedral angle at the centre, the dihedral angles of which are 80°, 105°, and 140°.

Ex. 808. The planes of the faces of a quadrangular spherical pyramid make with each other angles of 80°, 100°, 120°, and 150°, and a lateral edge of the pyramid is $3\frac{1}{2}$ feet. Find the volume of the pyramid.

Ex. 809. Having given the volume V, and the height h, of a spherical segment of one base, find the radius R of the sphere.

Ex. 810. Find the weight of a sphere of radius R, which floats in a liquid of specific gravity s, with one fourth of its surface above the surface of the liquid. (The weight of a floating body is equal to the weight of the liquid displaced.)

MISCELLANEOUS EXERCISES.

Ex. 811. Determine a point in a given plane such that the difference of its distances from two given points on opposite sides of the plane shall be a maximum.

Ex. 812. The portion of a tetrahedron cut off by a plane parallel to any face is a tetrahedron similar to the given tetrahedron.

Ex. 813. Two symmetrical tetrahedrons are equivalent.

Ex. 814. Two symmetrical polyhedrons may be decomposed into the same number of tetrahedrons symmetrical each to each.

Ex. 815. Two symmetrical polyhedrons are equivalent.

Ex. 816. If a solid has two planes of symmetry perpendicular to each other, the intersection of these planes is an axis of symmetry of the solid.

Ex. 817. If a solid has three planes of symmetry perpendicular to one another, the three intersections of these planes are three axes of symmetry of the solid; and the common intersection of these axes is the centre of symmetry of the solid.

Ex. 818. The volume of a sphere is to the volume of the inscribed cube as π is to $\frac{2}{3}\sqrt{3}$.

Ex. 819. Find the area of the surface of the sphere inscribed in a regular tetrahedron whose edge is 6 inches.

Ex. 820. If a zone of one base is the mean proportional between the remainder of the surface of the sphere and the total surface of the sphere, find the distance of the base of the zone from the centre of the sphere.

Ex. 821. Find the difference between the volume of a frustum of a pyramid and the volume of a prism each 24 feet high, if the bases of the frustum are squares with sides 20 feet and 16 feet, respectively, and the base of the prism is the section of the frustum parallel to the bases and midway between them.

Ex. 822. If the earth is assumed to be a sphere of 4000 miles radius, how far at sea can a lighthouse 100 feet high be seen?

Ex. 823. If the atmosphere extends 50 miles above the surface of the earth, and the earth is assumed to be a sphere of 4000 miles radius, find the volume of the atmosphere.

Ex. 824. Draw a line through the vertex of any trihedral angle, making equal angles with its edges.

Ex. 825. In any trihedral angle, the three planes passed through the edges and the respective bisectors of the opposite face angles intersect in the same straight line.

Ex. 826. In any trihedral angle, the three planes passed through the bisectors of the face angles, perpendicular to these faces, respectively, intersect in the same straight line.

Ex. 827. In any trihedral angle, the three planes passed through the edges, perpendicular to the opposite faces, respectively, intersect in the same straight line.

Ex. 828. In a tetrahedron, the planes passed through the three lateral edges and the middle points of the opposite sides of the base intersect in a straight line.

Ex. 829. The lines drawn from each vertex of a tetrahedron to the point of intersection of the medians of the opposite face all meet in a point called the *centre of gravity*, which divides each line so that the shorter segment is to the whole line in the ratio 1 : 4.

Ex. 830. The straight lines joining the middle points of the opposite edges of a tetrahedron all pass through the centre of gravity of the tetrahedron, and are bisected by the centre of gravity.

Ex. 831. The plane which bisects a dihedral angle of a tetrahedron divides the opposite edges into segments proportional to the areas of the faces that include the dihedral angle.

Ex. 832. The altitude of a regular tetrahedron is equal to the sum of the four perpendiculars let fall from any point within the tetrahedron upon the four faces.

Ex. 833. Within a given tetrahedron, to find a point such that the planes passed through this point and the edges of the tetrahedron shall divide the tetrahedron into four equivalent tetrahedrons.

Ex. 834. To cut a cube by a plane so that the section shall be a regular hexagon.

Ex. 835. Two tetrahedrons are similar if a dihedral angle of one is equal to a dihedral angle of the other, and the faces that include these angles are respectively similar, and similarly placed.

Ex. 836. To cut a tetrahedral angle so that the section shall be a parallelogram.

Ex. 837. Two polyhedrons composed of the same number of tetrahedrons, similar each to each and similarly placed, are similar.

Ex. 838. If the homologous faces of two similar pyramids are respectively parallel, the straight lines which join the homologous vertices of the pyramids meet in a point.

Ex. 839. The volume of a right circular cylinder is equal to the product of the lateral area by half the radius.

Ex. 840. The volume of a right circular cylinder is equal to the product of the area of the rectangle which generates it, by the length of the circumference generated by the point of intersection of the diagonals of the rectangle.

Ex. 841. If the altitude of a right circular cylinder is equal to the diameter of the base, the volume is equal to the total area multiplied by a third of the radius.

Ex. 842. Show that the prismatoid formula can be used for finding the volume of a sphere.

Ex. 843. Find the altitude of a zone equivalent to a great circle.

Ex. 844. Find the area of a spherical pentagon whose angles are 122°, 128°, 131°, 160°, 161°, if the surface of the sphere is 150 square feet.

Construct a spherical surface with given radius :

Ex. 845. Passing through a given point and tangent to two given planes.

Ex. 846. Passing through a given point and tangent to two given spheres.

Ex. 847. Passing through a given point and tangent to a given plane and a given sphere.

Ex. 848. Tangent to three given planes.

Ex. 849. Tangent to three given spheres.

Ex. 850. Tangent to two given planes and a given sphere.

Ex. 851. Tangent to two given spheres and a given plane.

Ex. 852. Through a given point to pass a plane tangent to a given circular cylinder.

Ex. 824. Draw a line through the vertex of any trihedral angle, making equal angles with its edges.

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Ex. 849. Tangent to three given spheres.

Ex. 850. Tangent to two given planes and a given sphere.

Ex. 851. Tangent to two given spheres and a given plane.

Ex. 852. Through a given point to pass a plane tangent to a given circular cylinder.

Ex. 853. Through a given point to pass a plane tangent to a given circular cone.

Ex. 854. Find the radius and the surface of a sphere whose volume is one cubic yard.

Ex. 855. Find the centre of a sphere whose surface passes through three given points, and touches a given plane.

Ex. 856. Find the centre of a sphere whose surface touches two given planes, and passes through two given points which lie between the planes.

Ex. 857. The volume of a sphere is two thirds the volume of the circumscribed circular cylinder, and its surface is two thirds the total surface of the cylinder.

Ex. 858. Given a sphere, a cylinder circumscribed about the sphere, and a cone of two nappes inscribed in the cylinder. If any two planes are drawn perpendicular to the axis of the three figures, the spherical segment between the planes is equivalent to the difference between the corresponding cylindrical and conic segments.

Ex. 859. A sphere 12 inches in diameter has an auger hole 3 inches in diameter through its centre. Find the remaining volume.

Ex. 860. Find the area of a solid generated by an equilateral triangle turning about one of its sides, if the length of the side is a.

Ex. 861. Compare the volumes of the solids generated by a rectangle turning successively about two adjacent sides, the lengths of these sides being a and b.

Ex. 862. An equilateral triangle revolves about one of its altitudes. Compare the lateral area of the cone generated by the triangle and the surface of the sphere generated by the inscribed circle.

Ex. 863. An equilateral triangle revolves about one of its altitudes. Compare the volumes of the solids generated by the triangle, the inscribed circle, and the circumscribed circle.

Ex. 864. The perpendicular let fall from the point of intersection of the medians of a given triangle upon any plane not cutting the triangle is equal to one third the sum of the perpendiculars from the vertices of the triangle upon the same plane.

Ex. 865. The perpendicular from the centre of gravity of a tetrahedron to a plane not cutting the tetrahedron is equal to one fourth the sum of the perpendiculars from the vertices of the tetrahedron to the plane.

BOOK IX.

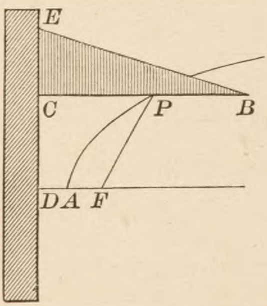
CONIC SECTIONS.

THE PARABOLA.

850. DEF. A **parabola** is a curve which is the locus of a point that moves in a plane so that its distance from a fixed point in the plane is always equal to its distance from a fixed line in the plane.

851. DEF. The fixed point is called the **focus**; and the fixed line, the **directrix**.

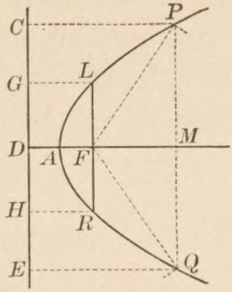
852. A parabola may be described by the continuous motion of a point, as follows:



Place a ruler so that one of its edges shall coincide with the directrix DE. Then place a right triangle with its base edge in contact with the edge of the ruler. Fasten one end of a string, whose length is equal to the other edge BC, to the point B, and the other end to a pin fixed at the focus F. Then slide the triangle BCE along the directrix, keeping the string tightly pressed against the ruler by the point of a pencil P. The point P will describe a parabola; for during the motion we always have PF equal to PC.

PROPOSITION I. PROBLEM.

853. *To construct a parabola by points, having given its focus and its directrix.*



Let F be the focus, and CDE the directrix.

Draw $FD \perp$ to CE, meeting CE at D. Bisect FD at A.

Then A is a point of the curve. § 850

Through any point M in the line DF, to the right of A, draw a line ∥ to CE.

With F as centre and DM as radius, draw arcs cutting this line at the points P and Q.

Then P and Q are points of the curve.

Proof. Draw PC and $QE \perp$ to CE.

Then $PC = DM$, and $QE = DM$, § 180

and $DM = PF = QF$. Const.

$\therefore PC = PF$, and $QE = QF$. Ax. 1

Therefore, P and Q are points of the curve. § 850

In this way any number of points may be found; and a continuous curve drawn through the points thus determined is the parabola whose focus is F and directrix CDE. Q.E.F.

854. Def. The point A is called the **vertex** of the curve. The line DF produced indefinitely in both directions is called the **axis** of the curve.

855. Def. The line FP, joining the focus to any point P on the curve, is called the **focal radius** of P.

856. Def. The distance AM is called the **abscissa**, and the distance PM the **ordinate**, of the point P.

857. Def. The double ordinate LR, through the focus, is called the **latus rectum** or **parameter**.

858. Cor. 1. *The parabola is symmetrical with respect to its axis.* § 210

For $FP = FQ$ (Const.), and, therefore, $PM = QM$. § 149

859. Cor. 2. *The curve lies entirely on one side of the perpendicular to the axis erected at the vertex; namely, on the same side as the focus.*

For any point on the other side of this perpendicular is obviously nearer to the directrix than to the focus.

860. Cor. 3. *The parabola is not a closed curve.*

For any point on the axis of the curve to the right of F is evidently nearer to the focus than to the directrix. Hence, the parabola QAP cannot cross the axis to the right of F.

861. Cor. 4. *The latus rectum is equal to* $4\,AF$.

For, if LG is drawn $\perp$ to CDE,

then $LF = LG$, and $LG = DF$. § 850

$\therefore LF = DF = 2\,AF$.

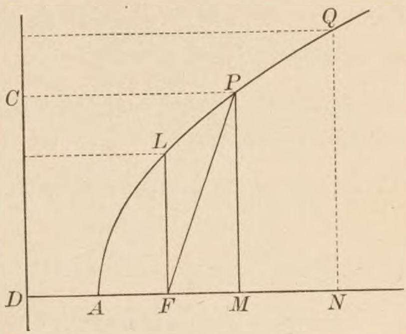
Similarly, $RF = DF = 2\,AF$.

Therefore, $LR = 4\,AF$.

Note. In the following propositions, the focus will be denoted by F, the vertex by A, and the point where the axis meets the directrix by D.

PROPOSITION II. THEOREM.

862. *The ordinate of any point of a parabola is the mean proportional between the latus rectum and the abscissa.*



Let P be any point, AM its abscissa, PM its ordinate.

To prove that $\overline{PM}^2 = 4\,AF \times AM$.

Proof. $$\overline{PM}^2 = \overline{FP}^2 - \overline{FM}^2 = \overline{DM}^2 - \overline{FM}^2 \qquad \S\ 850$$
$$= (DM - FM)(DM + FM)$$
$$= DF(DF + FM + FM)$$
$$= 2\,AF(2\,AF + 2\,FM)$$
$$= 2\,AF(2\,AM).$$

Hence, $$\overline{PM}^2 = 4\,AF \times AM.$$

Q. E. D.

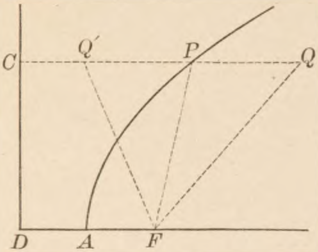
863. COR. 1. *The greater the abscissa of a point, the greater the ordinate.*

864. COR. 2. *The squares of any two ordinates are as the abscissas.*

For $$\frac{\overline{PM}^2}{\overline{QN}^2} = \frac{4\,AF \times AM}{4\,AF \times AN} = \frac{AM}{AN}.$$

PROPOSITION III. THEOREM.

865. *Every point within the parabola is nearer to the focus than to the directrix; and every point without the parabola is farther from the focus than from the directrix.*



1. **Let Q be a point within the parabola. Draw QC perpendicular to the directrix, cutting the curve at P. Draw QF and PF.**

To prove that $QF < QC.$

Proof. In the $\triangle QPF$, $QF < QP + PF.$ § 138

$\therefore QF < QP + PC.$

$\therefore QF < QC.$

2. **Let Q′ be a point without the curve. Draw Q′F.**

To prove that $Q'F > Q'C.$

Proof. In the $\triangle Q'FP$, $Q'F > PF - PQ'.$ § 138

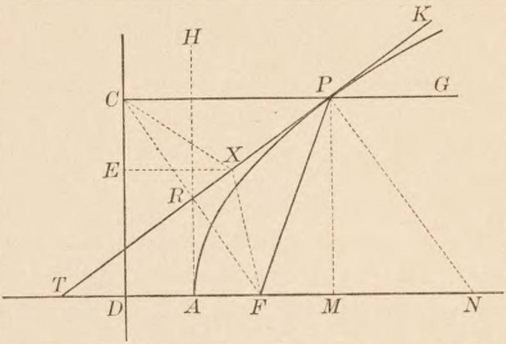
$\therefore Q'F > PC - PQ' > Q'C.$ Q. E. D.

866. COR. *A point is within or without a parabola according as its distance from the focus is less than, or greater than, its distance from the directrix.*

867. DEF. A straight line which touches, but does not cut, a parabola, is called a **tangent** to the parabola. The point where it touches the parabola is called the **point of contact.**

PROPOSITION IV. THEOREM.

868. *If a line PT is drawn from any point P of the curve, bisecting the angle between PF and the perpendicular from P to the directrix, every point of the line PT, except P, is without the curve.*



Let PC be the perpendicular from P to the directrix, the angle FPT equal the angle CPT, and let X be any other point in PT except P.

To prove that X is without the curve.

Proof. Draw $XE \perp$ to the directrix, and draw CX, FX, CF.

Let CF meet PT at R.

In the isos. $\triangle PCF$, $CR = RF$, § 149

and $CX = FX$. § 160

But $EX < CX$. § 97

$\therefore EX < FX$.

Hence, X is without the curve. § 866

Q. E. D.

869. COR. 1. *The bisector of the angle between PF and the perpendicular from P to the directrix is tangent to the curve at P.* § 867

870. COR. 2. *PT bisects FC, and is perpendicular to FC.*

871. Cor. 3. *Since the angles* FPT *and* FTP *are equal,* FT *equals* FP. § 147

872. Cor. 4. *The tangent at* A *is perpendicular to the axis.*

For it bisects the straight angle FAD.

873. Cor. 5. *The tangent at* A *is the locus of the foot of the perpendicular dropped from the focus to any tangent.*

Since $FR = RC$, and $FA = AD$, R is in AH. § 189

874. Def. The line PN drawn through P perpendicular to the tangent PT is called the **normal** at P.

875. Def. If the ordinate of P meets the axis in M, and the tangent and normal at P meet the axis in T and N respectively, then MT is the **subtangent** and MN the **subnormal.**

876. Cor. 6. *The subtangent is bisected by the vertex.*

For $FT = FP$ (§ 871), and $FP = DM$. § 850

Hence, $FT = DM$; also $AF = AD$.

Therefore, $FT - AF = DM - AD$. Ax. 3

That is, $TA = AM$.

877. Cor. 7. *The subnormal is equal to half the latus rectum.*

For $CP = FN$, and $CP = DM$. § 180

$\therefore FN = DM$ (Ax. 1); that is, $FM + MN = DF + FM$.

Therefore, $MN = DF$. Ax. 3

878. Cor. 8. *The normal bisects the angle between* FP *and* CP *produced; that is, bisects the angle* FPG.

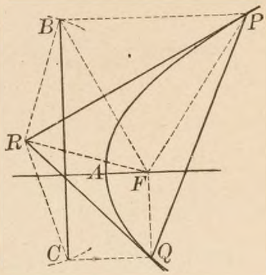
For $\angle NPT = \angle NPK$, and $\angle FPT = \angle TPC = \angle GPK$.

Hence, $\angle NPF = \angle NPG$. Ax. 3

879. Cor. 9. *The circle with* F *as centre and* FP *as radius passes through* T *and* N.

Proposition V. Problem.

880. *To draw a tangent to a parabola from an external point.*



Let R be any point external to the parabola QAP.

With R as centre and RF as radius, draw arcs cutting the directrix at the points B, C. Through B and C draw lines parallel to the axis, meeting the parabola in P, Q, respectively. Draw RP, RQ.

Then RP and RQ are tangents to the curve.

Proof. By construction, $RB = RF$, and $PB = PF$. § 850

Hence, $\angle RPB = \angle RPF$. § 150

Therefore, RP is the tangent at P. § 869

For like reason, RQ is the tangent at Q. Q. E. F.

881. Cor. *Two tangents can always be drawn to a parabola from an external point.*

For R is without the curve and nearer to the directrix than to the focus (§ 865); therefore, the circle with R as centre and RF as radius must always cut the directrix in two points.

882. Def. The line joining the points of contact P and Q is called the **chord of contact** for the tangents drawn from R.

Proposition VI. Theorem.

883. *The line joining the focus to the intersection of two tangents makes equal angles with the focal radii drawn to the points of contact.*

Let the tangents drawn at P and Q meet in R.

To prove that $\angle RFP = \angle RFQ$.

Proof. Draw the $\perp$s PB, QC to the directrix, and draw RB, RC, RF, FB.

	$\triangle RFP = \triangle RBP$.	§ 150
For	$PB = PF$,	§ 850
and	$RB = RF$.	§§ 870, 160
	$\therefore \angle RFP = \angle RBP$.	§ 128
Similarly,	$\angle RFQ = \angle RCQ$.	
Now	$\angle RBP = 90° + \angle RBC$,	
and	$\angle RCQ = 90° + \angle RCB$;	
and since	$RB = RF$, and $RC = RF$,	
therefore,	$RB = RC$.	Ax. 1
Hence,	$\angle RBC = \angle RCB$.	§ 145
Therefore,	$\angle RBP = \angle RCQ$,	
and	$\angle RFP = \angle RFQ$.	Q. E. D.

884. Cor. *The tangents drawn through the ends of a focal chord meet in the directrix.*

For, if the chord of contact PQ passes through F, then PFQ is a straight line.

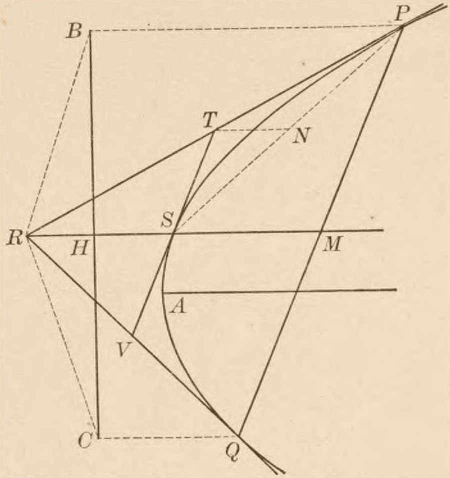
Hence, $\angle RFP + \angle RFQ = 180°$,

and $\angle RFP = \angle RFQ = 90°$.

Therefore, $\angle RBP = \angle RCQ = 90°$.

Proposition VII. Theorem.

885. *If two tangents RP and RQ are drawn from a point R to a parabola, the line drawn through R parallel to the axis bisects the chord of contact.*



Let the tangents drawn from R meet the curve in P, Q, and let the line through R parallel to the axis meet the directrix in H, the curve in S, and the chord of contact in M.

To prove that $PM = QM$.

Proof. Drop the ⊥s PB and QC to the directrix,

and draw RB, RC.

RH is ⊥ to BC, § 107

$RB = RC$. § 883

Hence, $BH = CH$. § 149

Now PB, QC, and MR are parallel. § 104

$\therefore PM = QM$. § 187

Q. E. D.

Proposition VIII. Theorem.

886. *If two tangents RP, RQ are drawn from a point R to a parabola, and through R a line parallel to the axis is drawn, meeting the curve in S, then the tangent at S is parallel to the chord of contact.*

Let the tangent at S meet the tangents PR, QR in T, V, respectively.

To prove that TV *is* $\parallel$ *to* PQ.

Proof. Draw $TN \parallel$ to SM, and let it meet SP in N.

Then	$PN = NS.$	§ 885
Hence,	$PT = TR.$	§ 188
Similarly,	$QV = VR.$	
Therefore,	TV is $\parallel$ to PQ.	§ 189

Q. E. D.

887. Cor. 1. *The line RM is the locus of the middle points of all chords drawn parallel to the tangent at S.*

For, if we suppose R to move along RM towards the curve, then since the point S and the direction of the tangent TV remain fixed, the chord PQ will remain parallel to TV, while its middle point M will move along MR towards S; finally, R, M, P, and Q will all coincide at S.

888. Def. The locus of the middle points of a system of parallel chords in a parabola is called a **diameter**. The parallel chords are called the **ordinates** of the diameter.

889. Cor. 2. *The diameters of a parabola are parallel to its axis; and conversely, every straight line parallel to the axis is a diameter; that is, bisects a system of parallel chords.*

890. Cor. 3. *Tangents drawn through the ends of an ordinate intersect in the diameter corresponding to that ordinate.*

891. Cor. 4. *The portion of a diameter contained between any ordinate and the intersection of the tangents drawn through the ends of the ordinate is bisected by the curve.*

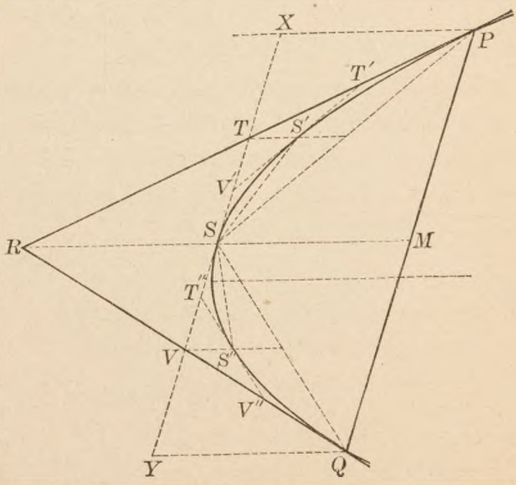
For the point S is the middle point of RM. § 188

892. Cor. 5. *The part of a tangent parallel to a chord contained between the two tangents drawn through the ends of the chord is bisected by the diameter of the chord at the point of contact.*

For the point S is also the middle point of the tangent TV.

Proposition IX. Theorem.

893. *The area of a parabolic segment made by a chord is two thirds the area of the triangle formed by the chord and the tangents drawn through the ends of the chord.*



Let PQ be any chord, and let the tangents at P and Q meet in R.

To prove that segment $PSQ \approx \frac{2}{3} \triangle PRQ$.

Proof. Draw the diameter RM, meeting the curve at S, and at S draw a tangent meeting PR in T and QR in V.

Draw SP, SQ.

Since $PT = TR$, and $QV = VR$, § 886

TV is $\parallel$ to PQ, § 189

and $PQ = 2 \times TV$. § 189

$\therefore \triangle PQS \approx 2 \triangle TVR$. § 405

If now we draw through T, V, the diameters TS', VS'', and then draw through S', S'', the tangents $T'S'V'$, $T''S''V''$, we can prove in the same way that

$$\triangle PSS' \approx 2 \triangle T'V'T,$$

and $$\triangle QSS'' \approx 2 \triangle T''V''V.$$

If we continue to form new triangles by drawing diameters through the points T', V', T'', V'', and tangents at the points where these diameters meet the curve, we can prove that each interior triangle formed by joining a point of contact to the extremities of a chord is twice as large as the exterior triangle formed by the tangents through these points, and hence that the sum of all the interior triangles is equal to twice the sum of the corresponding exterior triangles.

Now if we suppose the process to be continued indefinitely, then the limit of the sum of the interior triangles will be the segment PQS, and the limit of the sum of the exterior triangles will be the figure contained between the tangents PR, QR, and the curve.

But the sum of the interior triangles will always be equal to twice the sum of the exterior triangles; that is, to $\frac{2}{3}$ of the whole area, or $\frac{2}{3}$ the $\triangle PQR$.

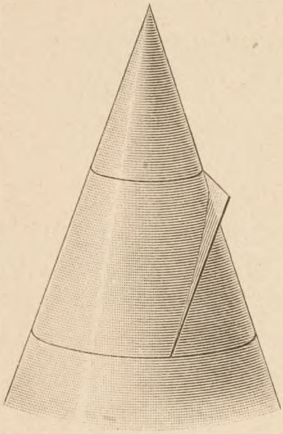
$\therefore$ segment $PQS \approx \frac{2}{3} \triangle PQR$. § 284

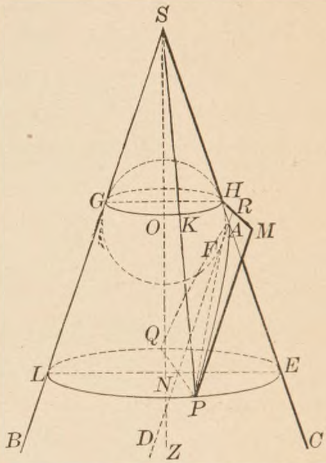
Q. E. D.

894. Cor. *If through* P *and* Q *lines are drawn parallel to* SM, *meeting the tangent* TV *produced in the points* X *and* Y, *then the segment* $PQS \approx \frac{2}{3} \square PQYX$.

PROPOSITION X. THEOREM.

895. *The section of a right circular cone made by a plane parallel to one, and only one, element of the surface is a parabola.*





Let SB be any element of the cone whose axis is SZ, and let QAP be the section of the cone made by a plane perpendicular to the plane BSZ and parallel to SB.

To prove that the curve PAQ is a parabola.

Proof. Let SC be the second element in which the plane BSZ cuts the cone, and let RAD be the intersection of the planes BSZ and PAQ.

Draw the $\odot$ O tangent to the lines SB, SC, RD, and let G, H, F be the points of contact, respectively.

Revolve BSC and the $\odot$ O about the axis SZ, the plane PAQ remaining fixed. The $\odot$ O will generate a sphere which will touch the cone in the $\odot$ GKH, and the plane PAQ at the point F.

Since SZ is $\perp$ to GH, SZ is $\perp$ to the plane GKH. § 501

Hence, the plane BSC is $\perp$ to the plane GKH. § 554

Let the plane of the $\odot$ GKH intersect the plane of the curve PAQ in the straight line MR; then will MR be $\perp$ to the plane BSC (§ 556), and therefore $\perp$ to DR.

Take any point P in the curve PAQ, and draw SP meeting the $\odot$ GH in K; draw FP, and draw PM $\perp$ to RM.

Pass a plane through P $\perp$ to the axis of the cone. Let it cut the cone in the $\odot$ $EPLQ$, and the plane of the curve PAQ in the line PNQ.

PN is $\perp$ to the plane BSC (§ 556), and therefore $\perp$ to DR.

Since PF and PK are tangents to the sphere O, they are tangents to the circle of the sphere made by a plane passing through the points P, F, K, and are therefore equal. § 261

That is, $PF = PK$.

But $PK = LG$. § 716

$\therefore PF = LG$. Ax. 1

Now LG and PM are each $\parallel$ to NR;

hence, LG is $\parallel$ to PM. § 521

The planes GKH and LPE are parallel. § 527

$\therefore LG = PM$. § 529

But $PF = LG$.

$\therefore PF = PM$. Ax. 1

That is, any point P on the curve PAQ is equidistant from a fixed point F and a fixed line RM in its plane.

Therefore, the curve PAQ is a parabola. § 850

Q. E. D.

EXERCISES.

Ex. 866. If the abscissa of a point is equal to its ordinate, each is equal to the latus rectum.

Ex. 867. If a secant PP' meets the directrix at H, then HF is the bisector of the exterior angle between the focal radii FP and FP'.

A straight line that cuts the curve is called a secant.

Ex. 868. To draw a tangent and a normal at a given point of a parabola.

Ex. 869. To draw a tangent to a parabola parallel to a given line.

Ex. 870. The tangents at the ends of the latus rectum meet at D.

Ex. 871. The latus rectum is the shortest focal chord.

Ex. 872. The tangent at any point meets the directrix and the latus rectum produced at points equally distant from the focus.

Ex. 873. The circle whose diameter is FP touches the tangent at A.

Ex. 874. The directrix touches the circle that has any focal chord for diameter.

Ex. 875. Given two points and the directrix, to find the focus.

Ex. 876. The perpendicular FC bisects TP. (See figure, page 414.)

Ex. 877. Given the focus and the axis, to describe a parabola which shall touch a given straight line.

Ex. 878. If PN is any normal, and the triangle PNF is equilateral, then PF is equal to the latus rectum.

Ex. 879. Given a parabola, to find the directrix, axis, and focus.

Ex. 880. To find the locus of the centre of a circle which passes through a given point and touches a given straight line.

Ex. 881. Given the axis, a tangent, and the point of contact, to find the focus and directrix.

Ex. 882. Given two points and the focus, to find the directrix.

Ex. 883. The triangles formed by the two tangents from any point and the focal radii to the points of contact are similar.

Ex. 884. If a diameter of a parabola is cut by a chord and the tangent at either end of the chord, the segments of the diameter between the tangent and the chord made by the curve are in the same ratio as the segments of the chord.

Ex = 18 3 +
4 −

THE ELLIPSE.

896. Def. An **ellipse** is a curve which is the locus of a point that moves in a plane so that the sum of its distances from two fixed points in the plane is constant.

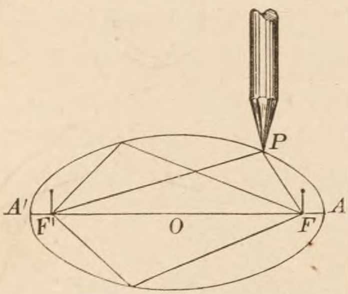
897. Def. The fixed points are called the **foci**, and the straight lines which join a point of the curve to the foci are called the **focal radii** of that point.

898. The constant sum of the focal radii is denoted by $2a$, and the distance between the foci by $2c$.

899. Def. The ratio $c : a$ is called the **eccentricity**, and is denoted by e. Therefore, $c = ae$.

900. Cor. *$2a$ must be greater than $2c$* (§ 138); *hence, e must be less than* 1.

901. The curve may be described by the continuous motion of a point, as follows:

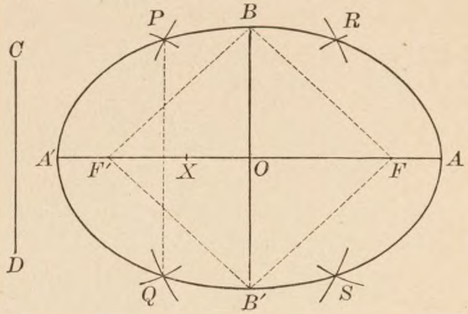


Fasten the ends of a string whose length is $2a$ at the foci F and F'. Trace a curve with the point P of a pencil pressed against the string so as to keep it stretched. The curve thus traced will be an ellipse whose foci are F and F', and the constant sum of whose focal radii is $FP + PF'$.

The curve is a closed curve extending around both foci; if A and A' are the points in which the curve cuts FF' produced, then AA' equals the length of the string.

PROPOSITION XI. PROBLEM.

902. *To construct an ellipse by points, having given the foci and the constant sum* $2a$.



Let F and F′ be the foci, and CD equal a.

Through the foci F, F' draw a straight line; bisect FF' at O. Lay off OA' equal to OA equal to CD.

Then A and A' are two points of the curve.

Proof. From the construction, $AA' = 2a$, and $AF = A'F'$.

Therefore, $AF + AF' = AF' + A'F' = AA' = 2a$,

and $A'F + A'F' = A'F + AF = AA' = 2a$.

To locate other points, mark any point X between F and F'. Describe arcs with F as centre and AX as radius; also other arcs with F' as centre and $A'X$ as radius; let these arcs cut in P and Q.

Then P and Q are two points of the curve.

This follows at once from the construction and § 896.

By describing the same arcs *with the foci interchanged*, two more points R, S may be found.

By assuming other points between F and F', and proceeding in the same way, any number of points may be found.

The curve passing through all the points is an ellipse having F and F' for foci, and $2a$ for the constant sum of focal radii.

Q. E. F.

903. Cor. 1. *By describing arcs from the foci with the same radius* OA, *we obtain two points* B, B' *of the curve which are equidistant from the foci. Therefore the line* BB' *is perpendicular to* AA' *and passes through* O. § 161

904. Def. The point O is called the **centre**. The line AA' is called the **major axis**; its ends A, A' are called the **vertices** of the curve. The line BB' is called the **minor axis**. The length of the minor axis is denoted by $2b$.

905. Cor. 2. *The major axis is bisected at* O, *and is equal to the constant sum* $2a$.

906. Cor. 3. *The minor axis is also bisected at* O. § 161

Therefore, $OB = OB' = b$.

907. Cor. 4. *The values of* a, b, c *are so related that*

$$a^2 = b^2 + c^2.$$

For, in the rt. $\triangle BOF$,

$$\overline{BF}^2 = \overline{OB}^2 + \overline{OF}^2. \qquad \S\ 371$$

908. Cor. 5. *The ellipse is symmetrical with respect to its major axis.*

For the axis AA' bisects PQ at right angles. § 161

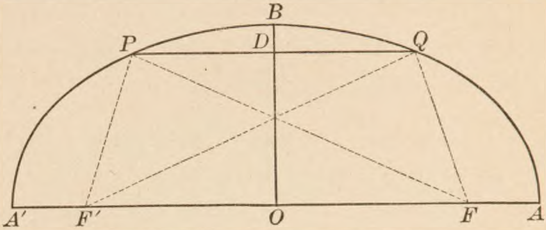
909. Def. The distance of a point of the curve from the minor axis is called the **abscissa** of the point, and its distance from the major axis is called the **ordinate** of the point.

910. Def. The double ordinate through the focus is called the **latus rectum** or **parameter**.

Note. In the following propositions F and F' denote the foci of the ellipse, O the centre, AA' the major axis, and BB' the minor axis.

Proposition XII. Theorem.

911. *An ellipse is symmetrical with respect to its minor axis.*



Let P be a point of the curve, PDQ be perpendicular to OB, meeting OB in D, and let DQ equal DP.

To prove that Q is also a point of the curve.

Proof. Join P and Q to the foci F, F'.

Revolve $ODQF$ about OD; F will fall on F', and Q on P.

Therefore,	$QF = PF'$,	
and	$\angle PQF = \angle QPF'$.	
Therefore,	$\triangle PQF = \triangle QPF'$,	§ 143
and	$QF' = PF$.	§ 128
Hence,	$QF + QF' = PF + PF'$.	Ax. 2
But	$PF + PF' = 2a$.	Hyp.
Therefore,	$QF + QF' = 2a$.	Ax. 1
Therefore,	Q is a point of the curve.	§ 896

Q.E.D.

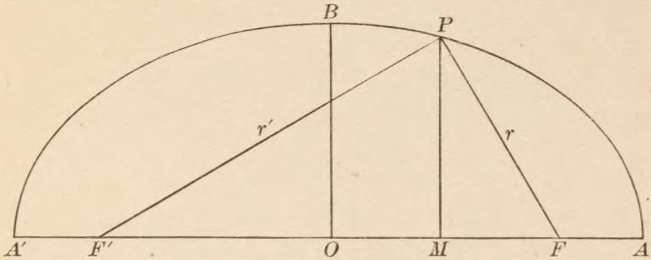
912. Def. Every chord that passes through the centre of an ellipse is called a **diameter**.

913. Cor. 1. *From* §§ 908, 911 *it follows that an ellipse consists of four equal quadrantal arcs symmetrically placed about the centre.* § 213

914. Cor. 2. *Every diameter is bisected at the centre.* § 209

PROPOSITION XIII. THEOREM.

915. *If d denotes the abscissa of a point of an ellipse, r and r' its focal radii, then $r' = a + ed$, $r = a - ed$.*



Let P be any point of an ellipse, PM perpendicular to AA′, d equal OM, r equal PF, r′ equal PF′.

To prove that $r' = a + ed$, $r = a - ed$.

Proof. From the rt. △ FPM and $F'PM$,

$$r^2 = \overline{PM}^2 + \overline{FM}^2,$$

and $$r'^2 = \overline{PM}^2 + \overline{F'M}^2. \qquad \S\ 371$$

Therefore, $$r'^2 - r^2 = \overline{F'M}^2 - \overline{FM}^2. \qquad \text{Ax. } 3$$

Or $(r' + r)(r' - r) = (F'M + FM)(F'M - FM)$.

Now $r' + r = 2a$, and $F'M + FM = 2c$.

Also, $F'M - FM = OF' + OM - FM = 2\,OM = 2d$.

Hence, $$2a(r' - r) = 4cd.$$

$$\therefore r' - r = \frac{2cd}{a} = 2ed.$$

From $r' + r = 2a$, and $r' - r = 2ed$,

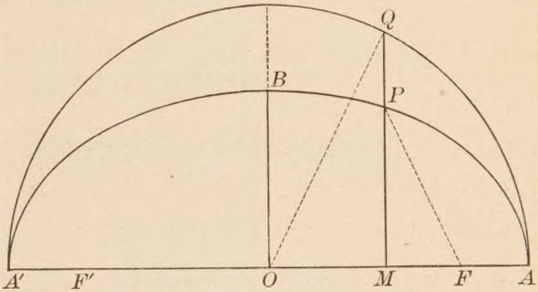
$$2r' = 2(a + ed), \text{ and } 2r = 2(a - ed).$$

Therefore, $r' = a + ed$, and $r = a - ed$. Q.E.D.

916. Def. The circle described upon the major axis of an ellipse as a diameter is called the **auxiliary circle.** The points where a line perpendicular to the major axis meets the ellipse and its auxiliary circle are called **corresponding points.**

Proposition XIV. Theorem.

917. *The ordinates of two corresponding points in an ellipse and its auxiliary circle are in the ratio $b : a$.*



Let P be a point of the ellipse, Q the corresponding point of the auxiliary circle, and QP meet AA′ at M.

To prove that $PM : QM = b : a$.

Proof. Let $OM = d$;

then $\overline{QM}^2 = a^2 - d^2$. § 372

Now $\overline{PM}^2 = \overline{PF}^2 - \overline{FM}^2 = (a - ed)^2 - (c - d)^2$ § 915

$$= a^2 - 2\,aed + e^2d^2 - c^2 + 2\,cd - d^2.$$

Or, since $c = ae$, and $a^2 - c^2 = b^2$, §§ 899, 907

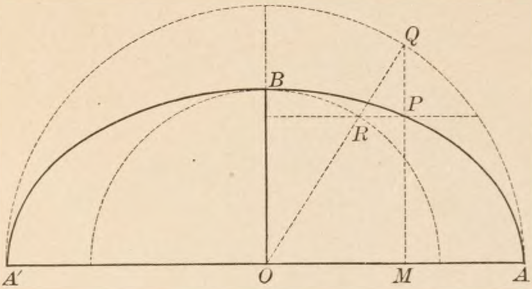
$$\overline{PM}^2 = b^2 - (1 - e^2)\,d^2 = \frac{b^2}{a^2}(a^2 - d^2).$$

Therefore, $\overline{PM}^2 : \overline{QM}^2 = b^2 : a^2$.

Or $PM : QM = b : a$. Q.E.D.

PROPOSITION XV. PROBLEM.

918. *To construct an ellipse by points, having given its two axes.*



Let OA, OB be the given semi-axes, O the centre.

With O as centre, and OA, OB, respectively, as radii, describe circles.

From O draw any straight line meeting the larger circle at Q and the smaller circle at R.

Through Q draw a line ∥ to BO, and through R draw a line ∥ to OA.

Let these lines meet at P.

Then will P be a point of the required ellipse.

Proof. If QP meet AA' at M,

$$PM : QM = OR : OQ. \qquad \S\ 343$$

But $OR = b$ and $OQ = a$.

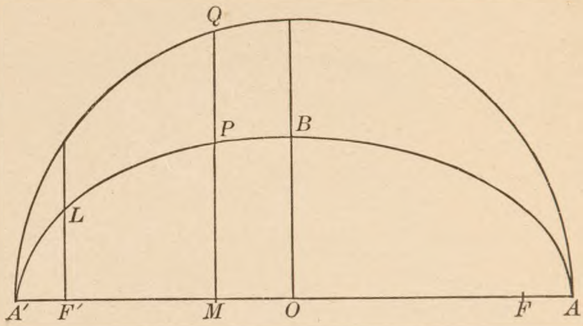
Therefore, $PM : QM = b : a$.

Therefore, P is a point of the ellipse. § 917

By drawing other lines through O, any number of points on the ellipse may be found; a smooth curve drawn through all the points will be the ellipse required. Q.E.F.

PROPOSITION XVI. THEOREM.

919. *The square of the ordinate of a point in an ellipse is to the product of the segments of the major axis made by the ordinate as* $b^2 : a^2$.



Let P, Q be corresponding points in the ellipse and auxiliary circle, respectively; let QP meet AA′ in M.

To prove that $\overline{PM}^2 : AM \times A'M = b^2 : a^2$.

Proof. $\overline{PM}^2 : \overline{QM}^2 = b^2 : a^2$. § 917

But $\overline{QM}^2 = AM \times A'M$. § 370

Therefore, $\overline{PM}^2 : AM \times A'M = b^2 : a^2$. Q.E.D.

920. COR. *The latus rectum is the third proportional to the major axis and the minor axis.*

For $\overline{LF'}^2 : AF' \times A'F' = b^2 : a^2$. § 919

Now $A'F' = a - c$, and $AF' = a + c$.

Therefore, $AF' \times A'F' = a^2 - c^2 = b^2$. § 907

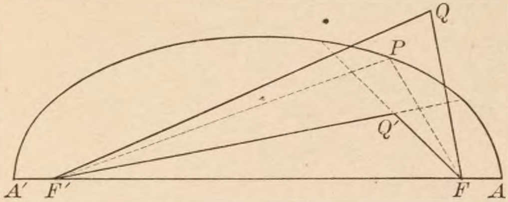
Hence, $\overline{LF'}^2 : b^2 = b^2 : a^2$,

and $LF' : b = b : a$.

Therefore, $2a : 2b = 2b : 2LF'$.

Proposition XVII. Theorem.

921. *The sum of the distances of any point from the foci of an ellipse is greater than or less than 2 a, according as the point is without or within the curve.*



1. **Let Q be a point without the curve.**

To prove that $QF + QF' > 2a.$

Proof. Let P be any point on the curve between QF and QF'. Draw PF and PF'.

Then $QF + QF' > PF + PF'$. § 100

But $PF + PF' = 2a$. § 896

Therefore, $QF + QF' > 2a$.

2. **Let Q′ be a point within the curve.**

To prove that $Q'F + Q'F' < 2a.$

Proof. Let P be any point of the curve included between FQ' produced and $F'Q'$ produced.

Then $Q'F + Q'F' < PF + PF'$. § 100

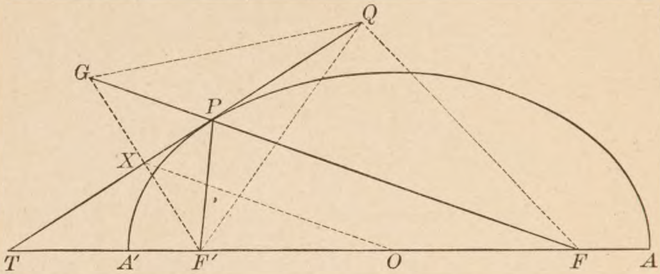
That is, $Q'F + Q'F' < 2a$. Q.E.D.

922. Cor. *Conversely, a point is without or within an ellipse according as the sum of its distances from the foci is greater than or less than 2 a.*

923. Def. A straight line which touches but does not cut an ellipse is called a **tangent** to the ellipse. The point where a tangent touches the ellipse is called the **point of contact.**

Proposition XVIII. Theorem.

924. *If through a point P of an ellipse a line is drawn bisecting the angle between one of the focal radii and the other produced, every point in this line except P is without the curve.*



Let PT bisect the angle F′PG between F′P and FP produced, and let Q be any point in PT except P.

To prove that Q is without the curve.

Proof. Upon FP produced take PG equal to PF'.

Draw GF', QF, QF', QG.

Then	$QG + QF > GF.$	§ 138
Now	$\triangle GPQ = \triangle F'PQ.$	§ 143
Therefore,	$QG = QF'.$	§ 128
Also	$GF = 2a.$	Const.
Therefore,	$QF' + QF > 2a.$	
Therefore,	Q is without the curve.	§ 922

Q. E. D.

925. Cor. 1. *The bisector of the angle between one of the focal radii from any point P and the other produced through P is a tangent to the curve at P.* § 923

926. Cor. 2. *The tangent to an ellipse at any point bisects the angle between one focal radius and the other produced.*

927. Cor. 3. *If GF' cuts PT at X, then $GX = F'X$, and PT is perpendicular to GF'.* § 161

928. Cor. 4. *The locus of the foot of a perpendicular from the focus of an ellipse to a tangent is the auxiliary circle.* This is called "Theorem of La Hire".

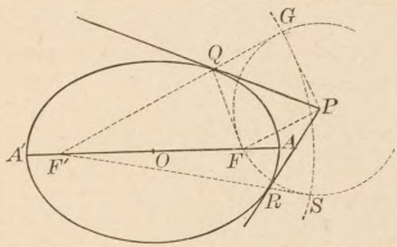
For $F'X = GX$, and $F'O = OF$.

Therefore, $OX = \frac{1}{2} FG = \frac{1}{2}(2a) = a$. § 189

Therefore, the point X lies in the auxiliary circle.

Proposition XIX. Problem.

929. *To draw a tangent to an ellipse from an external point.*



Let the arcs drawn with P as centre and PF as radius, and with F' as centre and $2a$ as radius intersect in G and S.

Draw GF' and SF', cutting the curve in Q and R, respectively.

Draw QP and RP, and they will be the tangents required.

Proof. By construction, $PG = PF$, and $QG = QF$. § 896

$\therefore \triangle PQG = \triangle PQF$. § 150

$\therefore \angle PQG = \angle PQF$. § 128

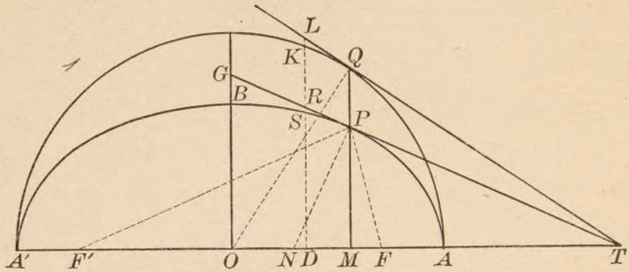
Therefore, PQ is the tangent at Q. § 925

For like reason PR is the tangent at R. Q.E.F.

930. Cor. *Two tangents may always be drawn to an ellipse from an external point.*

Proposition XX. Theorem.

931. *The tangents drawn at two corresponding points of an ellipse and its auxiliary circle cut the major axis produced at the same point.*



Let the tangent to the auxiliary circle at Q cut the major axis produced at T, and let the ordinate QM of the circle meet the ellipse at P. Draw TG through P.

To prove that TG is the tangent to the ellipse at P.

Proof. Through S, any point in the ellipse except P, draw $SD \perp$ to AA'; and let DS produced cut TG in R, the auxiliary circle in K, and the tangent at Q in L.

Then $RD : PM = DT : MT = LD : QM$, §§ 356, 351

or $RD : LD = PM : QM$. § 330

But $PM : QM = b : a$. § 917

$\therefore RD : LD = b : a$. Ax. 1

Again, $SD : KD = b : a$. § 917

$\therefore RD : LD = SD : KD$. Ax. 1

But $LD > KD$. Ax. 8

$\therefore RD > SD$.

$\therefore R$ is without the ellipse.

Hence, PT is the tangent at P. § 923

Q. E. D.

932. Cor. 1. $OT \times OM = a^2.$ § 367

933. Def. The straight line PN drawn through the point of contact of a tangent, perpendicular to the tangent, is called the **normal.**

934. Def. MT is called the **subtangent,** MN the **subnormal.**

935. Cor. 2. *The normal bisects the angle between the focal radii of the point of contact.*

For $\angle TPN = \angle GPN = 90°.$ § 933

But $\angle TPF = \angle GPF'.$ § 926

$\therefore \angle FPN = \angle F'PN.$ Ax. 3

Hence, a ray of light issuing from F will be reflected to F'.

936. Cor. 3. *If d denotes the abscissa of the point of contact, the distances measured on the major axis from the centre to the tangent and the normal are* $\frac{a^2}{d}$ *and* e^2d, *respectively.*

Since $OM = d$, and $OT \times OM = a^2$, § 932

therefore, $$OT = \frac{a^2}{d}.$$

Since $OM \times MT = \overline{QM}^2$, § 367

and $MN \times MT = \overline{PM}^2$,

therefore, $$\frac{OM}{MN} = \frac{\overline{QM}^2}{\overline{PM}^2} = \frac{a^2}{b^2}.$$

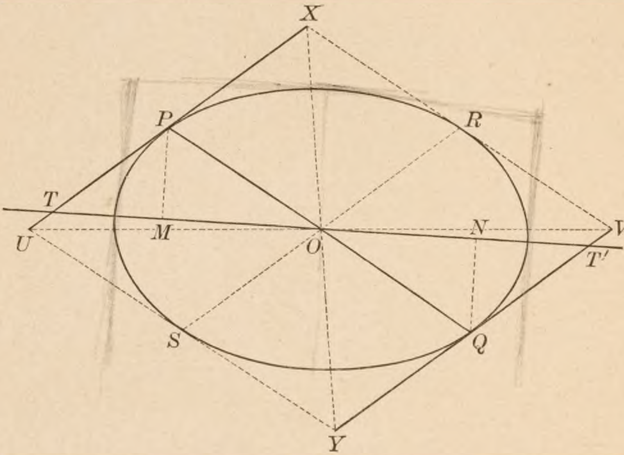
Therefore, $$\frac{OM - MN}{OM} = \frac{a^2 - b^2}{a^2} = \frac{c^2}{a^2} = e^2.$$ § 333

That is, $$\frac{ON}{OM} = e^2.$$

Hence, $$ON = e^2 \times OM = e^2d.$$

Proposition XXI. Theorem.

937. *The tangents drawn at the ends of any diameter are parallel to each other.*



Let POQ be any diameter, PT and QT′ the tangents at P, Q, respectively, meeting the major axis at T, T′.

To prove that PT is $\parallel$ to QT'.

Proof. Draw the ordinates PM, QN.

Then $\triangle OPM = \triangle OQN$ (§ 141), and $OM = ON$. § 128

But $OT = \dfrac{a^2}{OM}$, and $OT' = \dfrac{a^2}{ON}$ (§ 936). $\therefore OT = OT'$.

Therefore, $\triangle OPT = \triangle OQT'$, § 143

and $\angle OPT = \angle OQT'$. § 128

Hence, PT is $\parallel$ to QT'. § 111

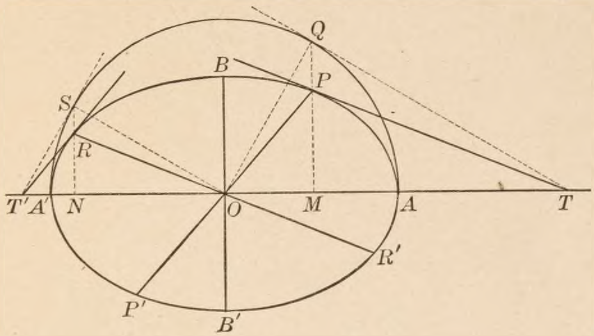
Q. E. D.

938. Def. One diameter is **conjugate** to another, if the first is parallel to the tangents at the extremities of the second.

Thus, if ROS is $\parallel$ to PT, RS is conjugate to PQ.

Proposition XXII. Theorem.

939. *If one diameter is conjugate to a second, the second is conjugate to the first.*



Let the diameter POP′ be parallel to the tangent RT′.

To prove that ROR' is parallel to the tangent PT.

Proof. Draw the ordinates PM and RN, and produce them to meet the auxiliary circle in Q and S, respectively.

Draw OP, OQ, OS; and draw the tangents QT, ST'.

Now, since OP is ∥ to RT',

	the $\triangle$ OMP and $T'NR$ are similar.	§ 354
	$\therefore T'N : OM = NR : MP.$	§ 351
But	$NR : NS = MP : MQ,$	§ 917
or	$NR : MP = NS : MQ.$	§ 330
	$\therefore T'N : OM = NS : MQ.$	Ax. 1
Hence,	$\triangle$ $T'NS$ and OMQ are similar.	§ 357
	$\therefore \angle NT'S = \angle MOQ.$	§ 351
	$\therefore T'S$ is ∥ to OQ.	§ 114
Hence,	$\angle QOS = \angle OST' = 90°.$	§ 254
	$\therefore SO$ is ∥ to QT.	§ 104

$\therefore \triangle s\ SNO$ and QMT are similar. § 354

$\therefore ON : TM = NS : MQ,$ § 351

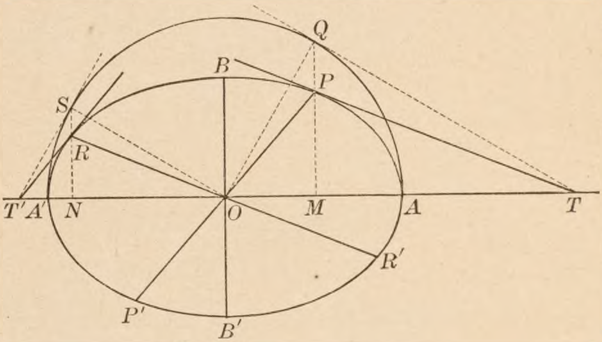
$= NR : MP.$ § 917

$\therefore \triangle s\ ONR$ and TMP are similar. § 357

$\therefore OR$ is $\parallel$ to PT. § 114

$\therefore RR'$ is conjugate to PP'. § 938

Q.E.D.



940. Cor. 1. *Angle* QOS *is a right angle.*

941. Cor. 2. $MP : ON = b : a$.

For $OS = OQ,$ § 217

and since $\angle NST' = \angle MQO,$ § 176

$\angle NSO = \angle MOQ.$ § 84

Hence, $\triangle NSO = \triangle MOQ.$ § 141

$\therefore ON = MQ.$ § 128

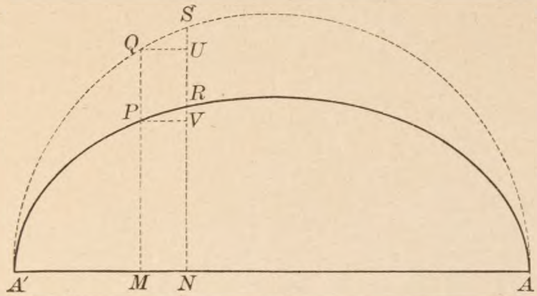
$\therefore MP : ON = MP : MQ.$

But $MP : MQ = b : a.$ § 917

Hence, $MP : ON = b : a.$ Ax. 1

Proposition XXIII. Theorem.

942. *The area of an ellipse is equal to πab.*



Let A′PRA be any semi-ellipse.

To prove that the area of twice $A'PRA$ is equal to πab.

Proof. Let PM, RN be two ordinates of the ellipse, and let Q, S be the corresponding points on the auxiliary circle.

Draw PV, $QU \parallel$ to the major axis, meeting NS in V, U.

Then the area of $\square PN = PM \times MN$, § 398

and the area of $\square QN = QM \times MN$. § 398

Therefore, $\dfrac{\square PN}{\square QN} = \dfrac{PM \times MN}{QM \times MN} = \dfrac{PM}{QM} = \dfrac{b}{a}$. § 917

The same relation will be true for all the rectangles that can be similarly drawn in the ellipse and auxiliary circle.

Hence, $\dfrac{\text{sum of} \boxed{s} \text{ in ellipse}}{\text{sum of} \boxed{s} \text{ in circle}} = \dfrac{b}{a}$. § 335

And this is true whatever be the number of the rectangles.

But the limit of the sum of the $\boxed{s}$ in the ellipse is the area of the ellipse, and the limit of those in the $\odot$ is the area of the $\odot$.

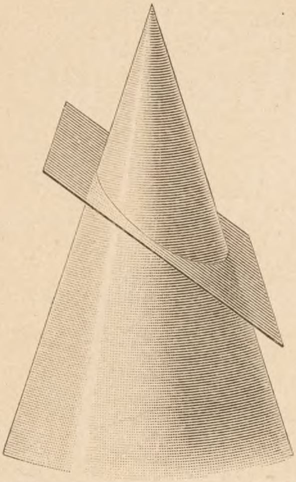
Therefore, $\dfrac{\text{area of ellipse}}{\text{area of circle}} = \dfrac{b}{a}$. § 285

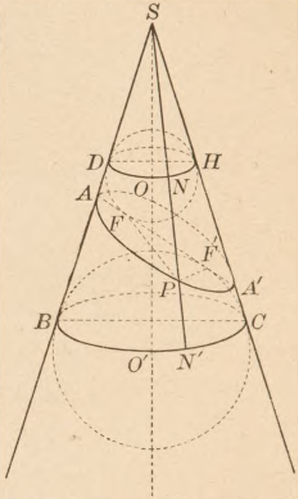
Therefore, the area of the ellipse $= \dfrac{b}{a} \times \pi a^2 = \pi ab$. § 463

Q.E.D.

Proposition XXIV. Theorem.

943. *The section of a right circular cone made by a plane that cuts all the elements of the surface of the cone is an ellipse.*





Let APA′ be the curve traced on the surface of the cone SBC by a plane that cuts all the elements of the surface of the cone.

To prove that the curve APA' is an ellipse.

Proof. The plane passed through the axis of the cone ⊥ to the secant plane APA' cuts the surface of the cone in the elements SB, SC, and the secant plane in the line AA'.

Describe the ⓢ O and O' tangent to SB, SC, AA'. Let the points of contact be D, H, F, and B, C, F', respectively.

Turn BSC and the ⓢ O, O' about the axis of the cone. The lines SB, SC will generate the surface of a right circular cone cut by the secant plane in the curve APA'; and the ⓢ O, O' will generate spheres which touch the cone in the ⓢ DNH, $BN'C$, and the secant plane in the points F, F'.

Let P be any point on the curve APA'. Draw PF, PF'; and draw SP, which touches the Ⓢ DH, BC at the points N, N', respectively.

Since PF and PN are tangent to the sphere O, they are tangent to the circle of the sphere made by a plane passing through P, F, and N.

Therefore, $PF = PN$. § 261

Likewise, $PF' = PN'$. § 261

Hence, $PF + PF' = PN + PN'$

$= NN'$, a constant quantity. § 716

Therefore, APA' is an ellipse with the points F and F' for foci, and AA' as $2a$. Q.E.D.

944. Cor. *If the secant plane is parallel to the base, the section is a circle.*

Ex. 885. The major axis is the longest chord that can be drawn in an ellipse.

Ex. 886. If the angle FBF' is a right angle, then $a^2 = 2b^2$.

Ex. 887. To draw a tangent and a normal at a given point of an ellipse.

Ex. 888. To draw a tangent to an ellipse parallel to a given straight line.

Ex. 889. Given the foci; to describe an ellipse touching a given straight line.

Ex. 890. Prove that $\overline{OF}^2 = OT \times ON$. (See figure, page 436.)

Ex. 891. Prove that $OM : ON = a^2 : c^2$. (See figure, page 436.)

Ex. 892. The minor axis is the shortest diameter of an ellipse.

Ex. 893. At what points of an ellipse will the normal pass through the centre of the ellipse?

Ex. 894. If FR, $F'S$ are the perpendiculars dropped from the foci to any tangent, then $FR \times F'S = b^2$.

Ex. 895. The semi-minor axis of an ellipse is the mean proportional between the segments of the major axis made by one of the foci.

Ex. 896. The area of an ellipse is to the area of its auxiliary circle as the minor axis is to the major axis.

Ex. 897. To draw a diameter conjugate to a given diameter in a given ellipse.

Ex. 898. Given $2a$, $2b$, one focus, and one point of an ellipse, to construct the ellipse.

Ex. 899. If from a point P a pair of tangents PQ and PR are drawn to an ellipse, then PQ and PR subtend equal angles at either focus.

Ex. 900. If a quadrilateral is circumscribed about an ellipse, either pair of its opposite sides subtend angles at either focus whose sum is equal to two right angles.

Ex. 901. To find the foci of an ellipse, having given the major axis and one point on the curve.

Ex. 902. To find the foci of an ellipse, having given the major axis and a straight line which touches the curve.

Ex. 903. If a straight line moves so that its extremities are always in contact with two fixed straight lines perpendicular to each other, then any point of the moving line describes an ellipse.

Ex. 904. To construct an ellipse, having given one of the foci and three tangents.

Ex. 905. To construct an ellipse, having given one focus, two tangents, and one of the points of contact.

Ex. 906. To construct an ellipse, having given one focus, one vertex, and one tangent.

Ex. 907. The area of the parallelogram formed by the tangents to an ellipse at the extremities of any pair of conjugate diameters is equal to the area of the rectangle contained by the axes of the ellipse.

Ex. 908. Given an ellipse, to find by construction the centre, the foci, and the axes.

Ex. 909. The circle described on any focal radius of an ellipse as a diameter is tangent to the auxiliary circle.

Ex. 910. If the ordinate and the tangent at any point P of an ellipse meet a diameter at H and K, respectively, then $OH \times OK = \overline{OQ}^2$. Q is the point in which the diameter cuts the curve.

THE HYPERBOLA.

945. Def. An **hyperbola** is a curve which is the locus of a point that moves in a plane so that the difference of its distances from two fixed points in the plane is constant.

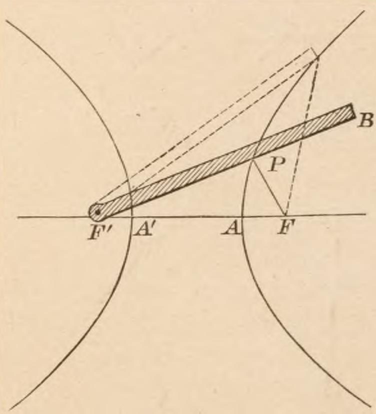
946. Def. The fixed points are called the **foci**, and the straight lines which join a point of the locus to the foci are called the **focal radii** of that point.

947. The constant difference of the focal radii is denoted by $2a$, and the distance between the foci by $2c$.

948. Def. The ratio $c:a$ is called the **eccentricity**, and is denoted by e. Therefore, $c = ae$.

949. Cor. *$2a$ must be less than $2c$* (§ 138); *hence, e must be greater than* 1.

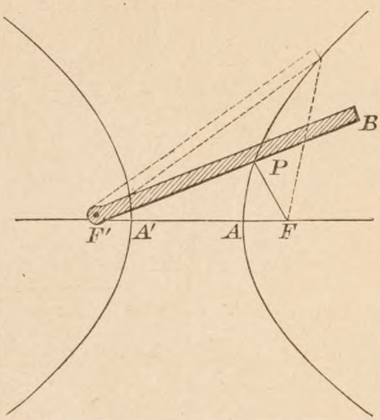
950. An hyperbola may be described by the continuous motion of a point, as follows:



To one of the foci F' fasten one end of a rigid bar $F'B$ so that it is capable of turning freely about F' as a centre in the plane of the paper.

Take a string whose length is less than that of the bar by the constant difference $2a$, and fasten one end of it at the other focus F, and the other end at the extremity B of the bar.

If now the rod is made to revolve about F' while the string is kept constantly stretched by the point of a pencil at P, in contact with the bar, the point P will trace an hyperbola.



For, as the bar revolves, $F'P$ and FP are each increasing by the same amount; namely, the length of that portion of the string which is removed from the bar between any two positions of P; hence, the difference between $F'P$ and FP will remain constantly the same.

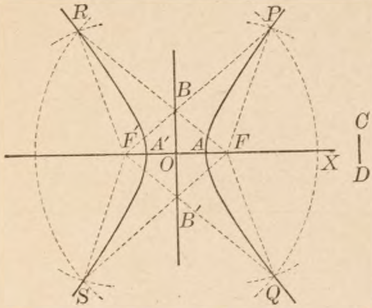
The curve obtained by turning the bar about F' is the right-hand *branch* of the hyperbola. Another similar branch on the left may be described in the same manner by making the bar revolve about F as a centre.

If the two branches of the hyperbola cut the line FF' at A and A', then, from the symmetry of the construction, $AA' = 2a$.

The hyperbola, therefore, consists of two similar branches which are separated at their nearest points by the distance $2a$, and which recede indefinitely from the line FF' and from one another.

PROPOSITION XXV. PROBLEM.

951. *To construct an hyperbola by points, having given the foci and the constant difference* $2a$.



Let F, F′ be the foci, and CD equal a.

Lay off OA equal to OA' equal to CD.

Then A and A' are two points of the curve.

Proof. From the construction, $AA' = 2a$ and $AF = A'F'$.

Therefore, $AF' - AF = AF' - A'F' = AA' = 2a$.

And $A'F - A'F' = A'F - AF = AA' = 2a$.

To locate other points, mark any point X in $F'F$ produced. Describe arcs with F' and F as centres, and $A'X$ and AX as radii, intersecting in P, Q.

Then P and Q are points of the curve.

By describing the same arcs with the foci interchanged, two more points R and S may be found.

By assuming other points in $F'F$ produced, any number of points may be found; and the curve passing through all these points is an hyperbola having F, F' for foci and $2a$ for the constant difference of the focal radii. Q.E.F.

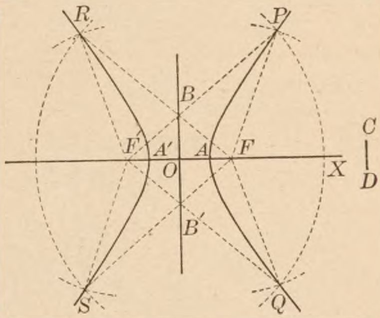
952. Cor. 1. *No point of the curve can be situated on the perpendicular to* FF′ *erected at* O.

For every point of this ⊥ is equidistant from the foci.

953. Def. The point O is called the **centre**; AA' is called the **transverse axis**; A and A' are called the **vertices.**

954. Def. In the ⊥ to FF' erected at O, let B, B' be two points at a distance from A (or A') equal to c; then BB' is called the **conjugate axis**, and is denoted by $2\,b$.

955. Def. If the transverse and conjugate axes are equal, the hyperbola is said to be **equilateral** or **rectangular.**



956. Cor. 2. *Both the axes are bisected at the centre.*

957. Cor. 3. *By* § 371, $c^2 = a^2 + b^2$.

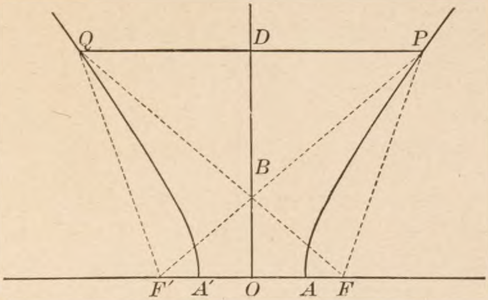
958. Cor. 4. *The curve is symmetrical with respect to the transverse axis.*

959. Def. The distances of a point of the curve from the transverse axis and the conjugate axis are called respectively the **ordinate** and **abscissa** of the point. The double ordinate through the focus is called the **latus rectum** or **parameter.**

Note. The letters A, A', B, B', F, F', and O will be used to designate the same points as in the above figure.

PROPOSITION XXVI. THEOREM.

960. *An hyperbola is symmetrical with respect to its conjugate axis.*



Let P be a point of the curve, PDQ be perpendicular to OB, meeting OB at D, and let DQ equal DP.

To prove that Q is also a point of the curve.

Proof. Join P and Q to the foci F, F'.

Turn $ODQF'$ about OD; F' will fall on F, and Q on P.

Therefore, $QF' = PF$, and $\angle PQF' = \angle QPF$.

Therefore, $\triangle PQF' = \triangle QPF$, § 143

and $QF = PF'$. § 128

Hence, $QF - QF' = PF' - PF$. Ax. 3

But $PF' - PF = 2a$. Hyp.

Therefore, $QF - QF' = 2a$. Ax. 1

Therefore, Q is a point of the curve. § 945

Q. E. D.

961. DEF. Every chord passing through the centre is called a diameter.

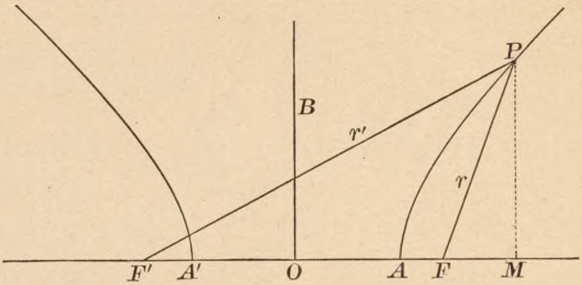
962. COR. 1. *An hyperbola consists of four equal quadrantal arcs symmetrically placed about the centre.* § 213

963. COR. 2. *Every diameter is bisected at the centre.* § 209

Proposition XXVII. Theorem.

964. *If d denotes the abscissa of a point of an hyperbola, r and r' its focal radii, then*

$$r = ed - a, \text{ and } r' = ed + a.$$



Let P be any point of the hyperbola, PM perpendicular to AA′, d equal OM, r equal PF, r′ equal PF′.

To prove that $r = ed - a,\ r' = ed + a.$

Proof. From the rt. △ FPM, $F'PM$,

$$r^2 = \overline{PM}^2 + \overline{FM}^2, \qquad \text{§ 371}$$

$$r'^2 = \overline{PM}^2 + \overline{F'M}^2.$$

Therefore, $r'^2 - r^2 = \overline{F'M}^2 - \overline{FM}^2.$

Or $(r' + r)(r' - r) = (F'M + FM)(F'M - FM).$

Now $r' - r = 2a$, and $F'M - FM = 2c.$

Also $F'M + FM = 2\,OF + 2\,FM = 2\,OM = 2d.$

By substitution, $a(r' + r) = 2cd.$

Or $$r' + r = \frac{2cd}{a} = 2ed.$$

From $r' + r = 2ed$, and $r' - r = 2a$,

by addition, $2r' = 2(ed + a)$;

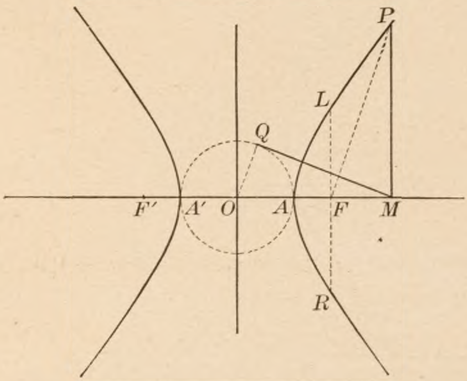
by subtraction, $2r = 2(ed - a)$.

Therefore, $r = ed - a$, and $r' = ed + a$. Q.E.D.

965. Def. The circle described upon AA' as a diameter is called the **auxiliary circle.**

PROPOSITION XXVIII. THEOREM.

966. *Any ordinate of an hyperbola is to the tangent from its foot to the auxiliary circle as b is to a.*



Let P be any point of the hyperbola, PM the ordinate, MQ the tangent drawn from M to the auxiliary circle.

To prove that $PM : QM = b : a.$

Proof. Let OM equal d.

Then $\overline{QM}^2 = d^2 - a^2.$ § 372

Also $\overline{PM}^2 = \overline{PF}^2 - \overline{FM}^2$

$= (ed - a)^2 - (d - c)^2$ § 964

$= e^2d^2 - 2\,aed + a^2 - d^2 + 2\,cd - c^2.$

Or since $c = ae$, and $a^2 - c^2 = -b^2$, §§ 948, 957

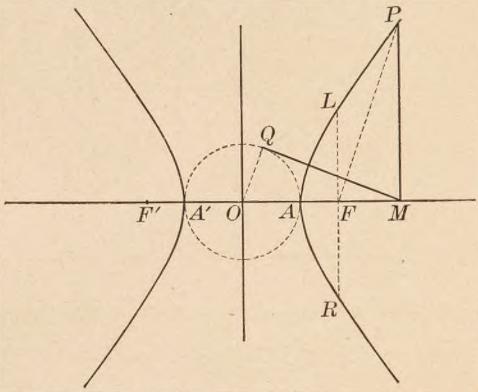
$$\overline{PM}^2 = (e^2 - 1)\,d^2 - b^2 = \frac{b^2}{a^2}(d^2 - a^2).$$

Therefore, $\overline{PM}^2 : \overline{QM}^2 = b^2 : a^2.$

Or $PM : QM = b : a.$ Q. E. D.

Proposition XXIX. Theorem.

967. *The square of the ordinate of a point in an hyperbola is to the product of the distances from the foot of the ordinate to the vertices as b^2 is to a^2.*



Let P be any point of the hyperbola, PM the ordinate, MQ the tangent from M to the auxiliary circle.

To prove that $\overline{PM}^2 : AM \times A'M = b^2 : a^2$.

Proof. Now $\overline{PM}^2 : \overline{QM}^2 = b^2 : a^2$. § 966

But $\overline{QM}^2 = AM \times A'M$. § 381

Therefore, $\overline{PM}^2 : AM \times A'M = b^2 : a^2$. Q. E. D.

968. Cor. *The latus rectum is the third proportional to the transverse and conjugate axes.*

For $\overline{LF}^2 : AF \times A'F = b^2 : a^2$. § 967

But $AF = c - a$, and $AF' = c + a$.

Therefore, $AF \times A'F = c^2 - a^2 = b^2$. § 957

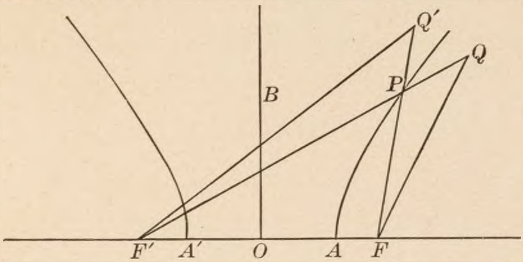
Hence, $\overline{LF}^2 : b^2 = b^2 : a^2$.

And $LF : b = b : a$.

Therefore, $2a : 2b = 2b : 2LF$.

PROPOSITION XXX. THEOREM.

969. *The difference of the distances of any point from the foci of an hyperbola is greater than or less than* $2a$*, according as the point is on the concave or convex side of the curve.*



1. **Let Q be a point on the concave side of the curve.**

To prove that $QF' - QF > 2a.$

Proof. Let QF' meet the curve at P.

$F'Q = F'P + PQ$ (Ax. 9), and $FQ < FP + PQ$. § 138

$\therefore F'Q - FQ > F'P - FP.$ Ax. 5

But $F'P - FP = 2a.$ § 947

Therefore, $F'Q - FQ > 2a.$

2. **Let Q′ be a point on the convex side of the curve.**

To prove that $Q'F' - Q'F < 2a.$

Proof. Let $Q'F$ cut the curve at P.

$F'Q' < F'P + PQ'$ (§ 138), and $FQ' = FP + PQ'$. Ax. 9

$\therefore F'Q' - FQ' < F'P - FP.$ Ax. 5

But $F'P - FP = 2a.$ § 947

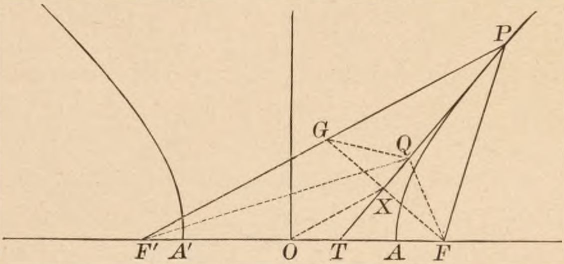
Therefore, $F'Q' - FQ' < 2a.$ Q.E.D.

970. COR. *Conversely, a point is on the concave or the convex side of the hyperbola according as the difference of its distances from the foci is greater than or less than* $2a$.

971. Def. A straight line which touches but does not cut the hyperbola is called a **tangent**, and the point where it touches the hyperbola is called the **point of contact to the hyperbola.**

Proposition XXXI. Theorem.

972. *If through a point P of an hyperbola a line is drawn bisecting the angle between the focal radii, every point in this line except P is on the convex side of the curve.*



Let PT bisect the angle FPF′, and let Q be any point in PT except P.

To prove that Q is on the convex side of the curve.

Proof. Take PG equal to PF; draw FG, QF, QF', QG.

Then $QF' - QG < GF'$. § 138

Also $\triangle PGQ = \triangle PFQ$ (§ 143); $\therefore QG = QF$.

Also $GF' = PF' - PF = 2a$.

Therefore, $QF' - QF < 2a$.

Therefore, Q is on the convex side of the curve. § 970

Q. E. D.

973. Cor. 1. *The bisector of the angle between the focal radii from any point P is the tangent to the curve at P.* § 971

974. Cor. 2. *The tangent to an hyperbola at any point bisects the angle between the focal radii drawn to that point.*

975. Cor. 3. *The tangent at A is perpendicular to AA′.*

976. Cor. 4. *If FG cuts PT at X, then $GX = FX$, and PT is perpendicular to FG.* § 161

977. Cor. 5. *The locus of the foot of the perpendicular from the focus to a tangent is the auxiliary circle.*

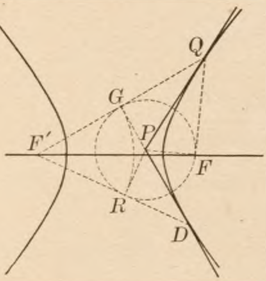
For $FX = GX$, and $FO = OF'$.

$\therefore OX = \frac{1}{2} F'G = \frac{1}{2}(PF' - PF) = a.$ § 189

Therefore, the point X lies on the auxiliary circle.

Proposition XXXII. Problem.

978. *To draw a tangent to an hyperbola from a given point P on the convex side of the hyperbola.*



Let the arcs described with P as centre and PF as radius, and with F' as centre and $2a$ as radius intersect in G and R.

Draw $F'G$ and $F'R$, and produce them to meet the curve in Q and D, respectively. Draw PQ and PD.

PQ and PD are the tangents required.

Proof. $PG = PF$, $QF = QF' - 2a = QG$.

$\therefore \triangle PQG = \triangle PQF.$ § 150

$\therefore \angle PQG = \angle PQF.$ § 128

$\therefore PQ$ is the tangent at Q. § 973

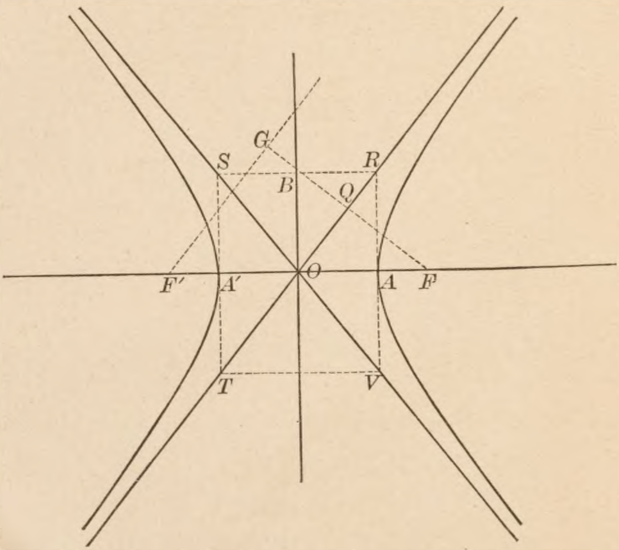
For like reason, PD is the tangent at D. Q.E.F.

979. Cor. *Two tangents may be drawn to an hyperbola from a point on the convex side of the hyperbola.*

980. Def. If a rectangle is constructed with its adjacent sides equal, respectively, to the transverse and conjugate axes of the hyperbola, and with one side tangent to the curve at A and its opposite side at A', its diagonals produced are called the **asymptotes of the hyperbola.**

Proposition XXXIII. Theorem.

981. *The asymptotes of an hyperbola never meet the curve, however far produced.*



Let TR be an asymptote of the hyperbola whose centre is O.

To prove that TR never meets the curve.

Proof. Let G be the intersection of arcs described from O and F' as centres with OF and $2a$, respectively, as radii.

Let TR meet the curve, if possible.

Draw FG, cutting TR at Q.

Then $OF' = OF$;

and $QG = QF$. § 976

$\therefore F'G$ is $\parallel$ to TR. § 189

Therefore, $F'G$ and TR cannot intersect. § 103

But if TR meets the curve, the point of contact must be at the intersection of $F'G$ and TR. § 978

Therefore, TR does not meet the curve. Q.E.D.

982. Cor. 1. *The line* FG *is tangent to the auxiliary circle at* Q.

For FG is $\perp$ to OR. § 976

Therefore, Q lies on the auxiliary circle. § 977

Hence, FG touches the auxiliary circle at Q. § 253

983. Cor. 2. FQ *is equal to the semi-conjugate axis* b.

For $\overline{FQ}^2 = \overline{OF}^2 - \overline{OQ}^2$, § 372

and $b^2 = c^2 - a^2$. § 957

But $OF = c$, and $OQ = a$.

Therefore, $FQ = b$.

984. Cor. 3. *If the tangent to the curve at* A *meets the asymptote* OR *at* R, *then* $AR = b$.

For $\triangle OAR = \triangle OQF$. § 142

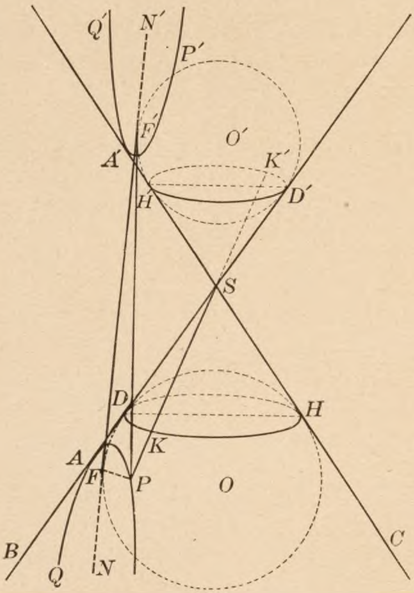
Therefore, $AR = FQ = b$.

985. Def. A perpendicular to a tangent erected at the point of contact is called a **normal**.

986. Def. The terms **subtangent** and **subnormal** are used in the hyperbola in the same sense as in the ellipse. § 934

Proposition XXXIV. Theorem.

987. *The section of a right circular cone made by a plane that cuts both nappes of the cone is an hyperbola.*



Let a plane cut the lower nappe of the cone in the curve PAQ, and the upper nappe in the curve P′A′Q′.

To prove that PAQ and $P'A'Q'$ are the two branches of an hyperbola.

Proof. The plane passed through the axis of the cone perpendicular to the secant plane cuts the surface of the cone in the elements BS, CS (prolonged through S), and the secant plane in the line NN'.

Describe the ⓢ O, O', tangent to BS, CS, NN'. Let the points of contact be D, H, F, and D', H', F', respectively.

Turn BSC and the Ⓢ O and O' about the axis of the cone. BS and CS will generate the surfaces of the two nappes of a right circular cone; and the Ⓢ O, O' will generate spheres which touch the cone in the Ⓢ DKH, $D'K'H'$, and the secant plane in the points F, F'.

Let P be any point on the curve. Draw PF and PF'; and draw PS, which touches the Ⓢ DKH, $D'K'H'$, at the points K, K'.

Now PF and PK are tangents to the sphere O from the point P.

Therefore, $PF = PK$. § 261

Also $PF' = PK'$. § 261

Hence, $PF' - PF = PK' - PK$ Ax. 3

$= KK'$, a constant quantity. § 716

Therefore, the curve is an hyperbola with the points F and F' for foci. § 945

Q. E. D.

TABLE OF FORMULAS.

PLANE FIGURES.

NOTATION.

$P =$ perimeter. $b =$ lower base.

$h =$ altitude. $b' =$ upper base.

$R =$ radius of circle.

$D =$ diameter of circle.

$C =$ circumference of circle.

$r =$ apothem of regular polygon.

$a, b, c =$ sides of triangle.

$s = \frac{1}{2}(a + b + c)$.

$p =$ perpendicular of triangle.

$m, n =$ segments of third side of triangle adjacent to sides b and a, respectively.

$S =$ area. $\pi = 3.1416$.

FORMULAS.

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POLYHEDRONS, CYLINDERS, AND CONES.

NOTATION.

S = lateral area. V = volume.

E = lateral edge; element. H = altitude.

P = perimeter of right section (Prisms and Cylinders).

p = perimeter of upper base. L = slant height.

P = perimeter of lower base. B = lower base.

c = circumference of upper base. b = upper base.

C = circumference of lower base. T = total area.

r = radius of upper base. M = area of mid-section.

R = radius of lower base.

a, b, c = dimensions of parallelopiped.

FORMULAS.

Prisms and Parallelopipeds.

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SPHERES.

NOTATION.

R = radius of sphere. D = diameter of sphere.

S = area of surface. H = altitude.

A = number of degrees in angle. B = base.

E = spherical excess. r, r' = radii of bases.

V = volume. T = sum of angles.

n = number of sides.

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